PRO-REPRESENTABILITY OF K^M-COHOMOLOGY IN WEIGHT 3 GENERALIZING A RESULT OF BLOCH

EOIN MACKALL

Abstract. We generalize a result, on the pro-representability of Milnor K-cohomology groups at the identity, that's due to Bloch. In particular, we prove, for X a smooth, proper, and geometrically connected variety defined over an algebraic field extension $k/\mathbb{Q}$, that the functor

$$\mathscr{T}_X^{i,3}(A) = \ker\left(\mathrm{H}^i(X_A, \mathcal{K}^M_{3,X_A}) \to \mathrm{H}^i(X, \mathcal{K}^M_{3,X})\right),$$

defined on Artin local k-algebras $(A, \mathfrak{m}_A)$ with $A/\mathfrak{m}_A \cong k$, is pro-representable provided that certain Hodge numbers of X vanish.

1. Introduction

Bloch [Blo75] has shown that the Milnor K-cohomology $\mathrm{H}^i(X, \mathcal{K}^M_{2,X})$, for a variety X defined over a number field k, is an interesting object amenable to study by deformation theory. More precisely, Bloch found a necessary and sufficient criterion for determining pro-representability of the functor which sends a local Artinian k-algebra A to the kernel of the canonical map

$$\mathrm{H}^i(X_A, \mathcal{K}^M_{2,X_A}) \to \mathrm{H}^i(X, \mathcal{K}^M_{2,X})$$

in terms of the vanishing of some Hodge numbers of X. Recall [Sch68] that a functor $F : \mathsf{Art}_k \to \mathsf{Set}$ from the category Art_k of local Artinian k-algebras $(A, \mathfrak{m}_A)$ with residue field $A/\mathfrak{m}_A \cong k$ is called *pro-representable* if there is a complete local Noetherian k-algebra $(R, \mathfrak{m}_R)$ such that:

(1) $(R/\mathfrak{m}_R^n, \mathfrak{m}_R/\mathfrak{m}_R^n)$ is an object in Art_k for all integers $n \geq 1$,
(2) there exists a canonical isomorphism of functors $F \simeq h_R$ between F and the functor $h_R : \mathsf{Art}_k \to \mathsf{Set}$ characterized by the assignment $h_R(A) = \mathrm{Hom}_{\mathrm{local}\ k-\mathrm{alg}}(R, A)$ for any object $(A, \mathfrak{m}_A)$ of Art_k.

A key ingredient necessary for the proof of Bloch's criterion is the existence of a certain logarithmic comparison isomorphism between the kernel of the restriction map $K_2(B) \to K_2(A)$, for a local $\mathbb{Q}$-algebra

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A and an A-algebra B with A-augmentation $\rho : B \to A$ so that the kernel $J = \ker(\rho)$ is nilpotent, and a certain subquotient of the group of absolute Kähler differentials for B (this was also proved in [Blo75]).

Since [Blo75], there have been some attempts at generalizing Bloch's results in different directions. To name some of the successes: Stienstra [Sti83] generalized the pro-representability criterion to surfaces defined over more general base fields; Maazen and Stienstra [MS78] provided an explicit presentation for $K_2(R)$ for a large class of rings R, allowing them to recover the relation to differentials provided by Bloch; and, more recently, Bloch's logarithmic comparison isomorphism has been extended to higher Milnor K-groups by both Gorchinskiy and Tyurin [GT18], and independently Dribus [Dri14].

In this note, we show how the theorems of Gorchinskiy and Tyurin, and Dribus, sheafify to allow study of pro-representability for functors that are related to the higher Milnor K-cohomology groups $\mathrm{H}^i(X, \mathcal{K}^M_{n,X})$. Our main result is a sufficiency criterion for pro-representability in the case of weight 3 Milnor K-cohomology. That is to say, we prove that for a variety X satisfying a host of assumptions (specifically, X should be very nice geometrically, defined over an algebraic extension $k/\mathbb{Q}$, and have some vanishing Hodge numbers), the functor which assigns to an Artin local k-algebra $(A, \mathfrak{m}_A)$ with residue field $A/\mathfrak{m}_A \cong k$ the group $\mathrm{H}^i(X_A, \mathcal{K}^M_{3,X_A})$ is pro-representable at the identity (Theorem 3.1). Our proof is also descriptive, like Bloch's, in the sense that we prove pro-representability by constructing an isomorphism

$$\ker\left(\mathrm{H}^i(X_A, \mathcal{K}^M_{3,X_A}) \to \mathrm{H}^i(X, \mathcal{K}^M_{3,X})\right) \cong \mathrm{H}^i(X, \Omega^2_{X/k}) \otimes_k \mathfrak{m}_A$$

which is of independent interest.

Lastly, we point out that most of the results proved here actually apply to all Milnor K-cohomology groups of arbitrary weight $n \geq 1$. Our main theorem is only limited to weight 3 because of the author's inability to determine the vanishing of certain sheaf cohomology groups related to differential p-forms for $p > 2$. So, if one could generalize the vanishing result of Lemma 3.3, then it should be possible to prove an appropriate sufficiency criterion for pro-representability in any weight.

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Conventions. We use the following conventions throughout:

- a variety is an integral scheme that is separated and of finite type over a base field.

Notation. We use the following notation throughout:

- $k/\mathbb{Q}$ is a fixed algebraic extension
- $(A, \mathfrak{m}_A)$ is an Artin local k-algebra with structure map $s : k \to A$ and residue field $A/\mathfrak{m}_A = k$
- X is a smooth k-variety with structure map $\phi : X \to \mathrm{Spec}(k)$ and $X_A = X \times_k \mathrm{Spec}(A)$ is the trivial deformation of X over $\mathrm{Spec}(A)$
- $\pi_1 : X_A \to X$ and $\pi_2 : X_A \to \mathrm{Spec}(A)$ are the first and second projections respectively.
- $\mathcal{J} \subset \mathcal{O}_{X_A}$ is the ideal sheaf of the reduction $\rho : X \to X_A$.

2. Generalizing Bloch's result

The purpose of this section is to collect preliminaries (mostly without proof) that directly generalize known results from the literature.

Definition 2.1. We write $\mathcal{R}^1_\rho$ for the kernel sheaf

$$\mathcal{R}^1_\rho = \ker\left(\Omega^1_{X_A/k} \twoheadrightarrow \rho_*\Omega^1_{X/k}\right).$$

This sheaf is sometimes denoted $\Omega^1_{X_A,\mathcal{J}}$ or $\Omega^1_{X\times A, X\times \mathfrak{m}_A}$ in the literature. Note that $\mathcal{R}^1_\rho$ is coherent.

Definition 2.2. For any $n > 1$ we define $\mathcal{R}^n_\rho$ as the kernel

$$\mathcal{R}^n_\rho = \ker\left(\Omega^n_{X_A/k} \twoheadrightarrow \wedge^n\left(\rho_*\Omega^1_{X/k}\right)\right).$$

This is equal to the image sheaf

$$\mathcal{R}^n_\rho = \mathrm{Im}\left(\mathcal{R}^1_\rho \otimes_{\mathcal{O}_{X_A}} \Omega^{n-1}_{X_A/k} \twoheadrightarrow \Omega^n_{X_A/k}\right).$$

Note that $\mathcal{R}^n_\rho$ is also coherent.

Remark 2.3. The differentials $d^n : \Omega^n_{X_A/k} \to \Omega^{n+1}_{X_A/k}$ restricted to the subsheaves $\mathcal{R}^n_\rho$ fit together in a complex

$$\mathcal{R}^0_\rho := \mathcal{J} \to \mathcal{R}^1_\rho \xrightarrow{d^1} \mathcal{R}^2_\rho \xrightarrow{d^2} \cdots \xrightarrow{d^{i-1}} \mathcal{R}^i_\rho \xrightarrow{d^i} \cdots.$$

We will often write d to mean any one of these differentials.

For any scheme Y we write $\mathcal{K}^M_{n,Y}$ for the Zariski sheaf associated to the presheaf of Milnor K-groups defined by the assignment

$$U \rightsquigarrow K^M_n(\mathcal{O}_Y(U))$$

for any open $U \subset Y$. There is then an exact sequence

$$0 \to \mathcal{K}^M_{n,\rho} \to \mathcal{K}^M_{n,X_A} \to \mathcal{K}^M_{n,X} \to 0$$

induced by the reduction $\mathcal{O}_{X_A} \to \rho_*\mathcal{O}_X$ and with $\mathcal{K}^M_{n,\rho}$ defined to be the appropriate kernel sheaf. This exact sequence is, moreover, right-split by the structure map

$$\mathcal{O}_X \to s_*\mathcal{O}_{X_A} \to s_*\rho_*\mathcal{O}_X = \mathcal{O}_X.$$

Lemma 2.4. *For each $n \geq 2$, there is an isomorphism of sheaves of abelian groups*

$$\psi^n : \mathcal{K}^M_{n,\rho} \xrightarrow{\sim} \mathcal{R}^{n-1}_\rho / d\mathcal{R}^{n-2}_\rho$$

coming from sheafifying the Bloch maps of [GT18, Theorem 2.10]. □

From now on we identify $\Omega^1_{X_A/k}$ with the sum

$$\Omega^1_{X_A/k} \cong \pi_1^*\Omega^1_{X/k} \oplus \pi_2^*\Omega^1_{A/k}$$

via the isomorphism of [Sta21, Tag 01V1]. Similarly we identify

$$\Omega^n_{X_A/k} \cong \bigoplus_{j=0}^{n} \Omega^j_{X/k} \boxtimes \Omega^{n-j}_{A/k} \tag{Kün}$$

where for any $0 \leq j \leq n$ we use the notation

$$\Omega^j_{X/k} \boxtimes \Omega^{n-j}_{A/k} := \pi_1^*\Omega^j_{X/k} \otimes_{\mathcal{O}_{X_A}} \pi_2^*\Omega^{n-j}_{A/k} \quad \left(\cong \Omega^j_{X/k} \otimes_k \Omega^{n-j}_{A/k}\right)$$

for the exterior product of sheaves on the product $X_A = X \times_k \mathrm{Spec}(A)$. The following two lemmas are direct generalizations from the case when $n = 1$ which is observed in [Blo75, §3].

Lemma 2.5. *The composition*

$$\mathcal{R}^n_\rho \to \Omega^n_{X_A/k} \twoheadrightarrow \bigoplus_{j=0}^{n-1} \Omega^j_{X/k} \boxtimes \Omega^{n-j}_{A/k}$$

of the natural inclusion and projection induces a short exact sequence

$$0 \to \Omega^n_{X/k} \boxtimes \mathfrak{m}_A \to \mathcal{R}^n_\rho \to \bigoplus_{j=0}^{n-1} \Omega^j_{X/k} \boxtimes \Omega^{n-j}_{A/k} \to 0.$$

This exact sequence is, moreover, split. □

From now on we write $\mathfrak{m}^0_A = \ker(\mathfrak{m}_A \xrightarrow{d} \Omega^1_{A/k})$ for the given kernel.

Lemma 2.6. *For any $n \geq 1$, the following square of differentials and inclusions is commutative.*

$$\begin{array}{ccc} \Omega^{n-1}_{X/k} \otimes_k \mathfrak{m}^0_A & \longrightarrow & \mathcal{R}^{n-1}_\rho \\ \downarrow{\scriptstyle d\otimes 1} & & \downarrow{\scriptstyle d} \\ \Omega^n_{X/k} \boxtimes \mathfrak{m}_A & \longrightarrow & \mathcal{R}^n_\rho \end{array} \tag{$*$}$$

From the square ($*$) *one also gets an exact sequence*

$$0 \to \frac{\Omega^n_{X/k} \boxtimes \mathfrak{m}_A}{\operatorname{Im}(d \otimes 1)} \to \frac{\mathcal{R}^n_\rho}{d\mathcal{R}^{n-1}_\rho} \to \mathcal{C}^n_\rho \to 0$$

where

$$\mathcal{C}^n_\rho = \frac{\bigoplus_{j=0}^{n-1} \Omega^j_{X/k} \boxtimes \Omega^{n-j}_{A/k}}{\Omega^{n-1}_{X/k} \otimes_k d(\mathfrak{m}_A) + d\left(\bigoplus_{j=0}^{n-2} \Omega^j_{X/k} \boxtimes \Omega^{n-1-j}_{A/k}\right)}$$

is the canonical quotient sheaf.

Proof. To see that the square ($*$) above is commutative, it suffices to observe that the differential d of the de Rham complex $\Omega^\bullet_{X_A/k}$ is identifiable, via the isomorphism of (Kün), with the differential of the total complex $\operatorname{Tot}(\Omega^\bullet_{X/k} \boxtimes \Omega^\bullet_{A/k})$; for more information, one can consult [Sta21, Tag 0FM9,Tag 012Z]. It follows from the existence of ($*$) that there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^{n-1}_{X/k} \otimes_k \mathfrak{m}^0_A & \longrightarrow & \mathcal{R}^{n-1}_\rho & \longrightarrow & \mathcal{E} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d\otimes 1} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle \tilde{d}} & & \\
0 & \longrightarrow & \Omega^n_{X/k} \boxtimes \mathfrak{m}_A & \longrightarrow & \mathcal{R}^n_\rho & \longrightarrow & \bigoplus_{j=0}^{n-1} \Omega^j_{X/k} \boxtimes \Omega^{n-j}_{A/k} & \longrightarrow & 0
\end{array}$$

where we use the placeholder

$$\mathcal{E} = \Omega^{n-1}_{X/k} \otimes_k \left(\mathfrak{m}_A/\mathfrak{m}^0_A\right) \oplus \bigoplus_{j=0}^{n-2} \Omega^j_{X/k} \boxtimes \Omega^{n-1-j}_{A/k}$$

and write $\tilde{d}$ for the induced map forming the rightmost vertical arrow.

We prove in Lemma 2.7 below that the kernel of the composition

$$\mathcal{R}^{n-1}_\rho \xrightarrow{d} \mathcal{R}^n_\rho \to \bigoplus_{j=0}^{n-1} \Omega^j_{X/k} \boxtimes \Omega^{n-j}_{A/k}$$

is locally generated by both $\ker(d)$ and $\Omega^{n-1}_{X/k} \otimes_k \mathfrak{m}^0_A$. This implies that, for each point $x \in X_A$, any local section ψ of $\mathcal{E}_x$ in the kernel of $\tilde{d}$ can be lifted to a local section $\psi' \in (\mathcal{R}^{n-1}_\rho)_x$ that can be written as a sum of local sections from $\ker(d)_x$. The final claim of the lemma, regarding the given exact sequence, then follows from an application of the snake lemma to the above commutative diagram. □

Lemma 2.7. *With notation as in Lemma 2.6, let $x \in X_A$ be a point and let $\psi \in (\mathcal{R}^{n-1}_\rho)_x$ be a local section such that $d(\psi)$ is contained in $(\Omega^n_{X/k} \boxtimes \mathfrak{m}_A)_x \subset (\mathcal{R}^n_\rho)_x$. Then $\psi \in \ker(d)_x + (\Omega^{n-1}_{X/k} \otimes_k \mathfrak{m}^0_A)_x$.*

Proof. Fix a basis $\{e_i\}_{i\in I}$ for the k-vector space $\mathfrak{m}_A^0$ and extend this to a basis $\{e_i\}_{i\in J}$ for the k-vector space $\mathfrak{m}_A$. We note that it's possible to have $\mathfrak{m}_A^0 = 0$, in which case we have $I = \emptyset$, and we can assume that $\mathfrak{m}_A \neq 0$ since if $\mathfrak{m}_A = 0$ then $A = k$ and the claim is immediate.

By general properties of Kähler differentials, in particular [Sta21, Tag 02HP], we have that $\Omega^1_{A/k}$ is generated as an A-module by the elements $\{de_i\}_{i\in J}$. Considering the A-module $\Omega^1_{A/k}$ as a k-vector space, it follows that the elements

$$\{de_i\}_{i\in J\setminus I}, \quad \text{and} \quad \{e_j de_i\}_{j\in J, i\in J\setminus I}$$

span all of $\Omega^1_{A/k}$. The elements $\{de_i\}_{i\in J\setminus I}$ are k-linearly independent by construction so, by restricting to a subset $J' \subset J$, we can suppose that the collection

$$\{de_i\}_{i\in J\setminus I} \cup \{e_j de_i\}_{j\in J', i\in J\setminus I}$$

forms a basis for $\Omega^1_{A/k}$.

Using the splitting of $\mathcal{R}^{n-1}_\rho$ given in Lemma 2.5, an arbitrary local section $\psi \in (\mathcal{R}^{n-1}_\rho)_x$ can be written as a sum

$$\psi = \sum_{i\in I} \omega_i \otimes e_i + \sum_{j\in J\setminus I} \omega_j \otimes e_j + \sum_{r\in J\setminus I} \alpha_r \otimes de_r + \sum_{r\in J\setminus I, s\in J'} \alpha_{r,s} \otimes e_s de_r + \psi'$$

where we have:

- $\omega_i \in \Omega^{n-1}_{X/k,x}$ for all $i \in I$,
- $\omega_j \in \Omega^{n-1}_{X/k,x}$ for all $j \in J \setminus I$,
- $\alpha_r \in \Omega^{n-2}_{X/k,x}$ for all $r \in J \setminus I$,
- $\alpha_{r,s} \in \Omega^{n-2}_{X/k,x}$ for all pairs $(r,s) \in (J\setminus I) \times J'$,
- and for a uniquely determined local section ψ' of $(\mathcal{R}^{n-1}_\rho)_x$ contained in the summand coming from $\bigoplus_{j=2}^{n-1} \Omega^{n-1-j}_{X/k} \boxtimes \Omega^j_{A/k}$.

Applying the differential d to the local section ψ then gives:

$$\begin{aligned} d(\psi) = &\sum_{i\in I} d\omega_i \otimes e_i \\ &+ \sum_{j\in J\setminus I} d\omega_j \otimes e_j + (-1)^{n-1}\omega_j \otimes de_j \\ &+ \sum_{r\in J\setminus I} d\alpha_r \otimes de_r \\ &+ \sum_{r\in J\setminus I, s\in J'} d(\alpha_{r,s}) \otimes e_s de_r + (-1)^{n-2}\alpha_{r,s} \otimes de_s \wedge de_r \\ &+ d\psi'. \end{aligned}$$

Combining terms by their appropriate bidegrees, we can write:

$$d(\psi) - \psi'' = \sum_{i\in J} d\omega_i \otimes e_i + \sum_{j\in J\setminus I} (d\alpha_j + (-1)^{n-1}\omega_j) \otimes de_j + \sum_{r\in J\setminus I, s\in J'} d(\alpha_{r,s}) \otimes e_s de_r$$

for a local section ψ'' of $(\mathcal{R}^n_\rho)_x$ contained in the sum of those summands where $\Omega^j_{A/k}$ appears as a factor for any $j \geq 2$.

Now suppose that $d(\psi)$ is contained in $(\Omega^n_{X/k} \boxtimes \mathfrak{m}_A)_x$. It follows that $\psi'' = 0$, there is vanishing $d(\alpha_{r,s}) = 0$ for all $r \in J \setminus I$, and we get an equality $d\alpha_j + (-1)^{n-1}\omega_j = 0$ for all $j \in J \setminus I$. From the latter of these we find

$$\omega_j = d((-1)^n \alpha_j)$$

for all $j \in J \setminus I$. From this we find that both

$$\sum_{j\in J\setminus I} \omega_j \otimes e_j + \sum_{r\in J\setminus I} \alpha_r \otimes de_r \quad \text{and} \quad \sum_{r\in J\setminus I, s\in J'} \alpha_{r,s} \otimes e_s de_r + \psi'$$

are elements of $\ker(d)_x$. This immediately implies the claim, comparing with the presentation of ψ above. □

Example 2.8. If $A = k[\epsilon]/(\epsilon^2)$ is the ring of dual numbers, then both $\mathfrak{m}^0_A = 0$ and $\mathcal{C}^n_\rho = 0$. It follows that

$$\mathrm{H}^i(X, \mathcal{R}^n_\rho / d\mathcal{R}^{n-1}_\rho) = \mathrm{H}^i(X, \Omega^n_{X/k})$$

for any $i \geq 0$ and for any $n \geq 1$ in this case.

Example 2.9. In the case $n = 1$, the sheaf $\mathcal{C}^n_\rho$ simplifies to

$$\mathcal{C}^1_\rho = \mathcal{O}_X \otimes_k \left(\Omega^1_{A/k}/d(\mathfrak{m}_A)\right).$$

This is the case that's studied in [Blo75]; explicitly, the sheaf $\mathcal{C}^1_\rho$ appears in [Blo75, (3.2)].

Example 2.10. For $n = 2$ the above sheaf $\mathcal{C}^n_\rho$ becomes

$$\mathcal{C}^2_\rho = \frac{\left(\Omega^1_{X/k} \boxtimes \Omega^1_{A/k}\right) \oplus \left(\mathcal{O}_X \boxtimes \Omega^2_{A/k}\right)}{\Omega^1_{X/k} \otimes_k d(\mathfrak{m}_A) + d(\mathcal{O}_X \boxtimes \Omega^1_{A/k})}.$$

We study this sheaf in more detail in the next section.

3. Weight 3

We write $\mathscr{T}_X^{i,n}$ for the functor which assigns to an Artin local k-algebra $(A, \mathfrak{m}_A)$ with residue field $A/\mathfrak{m}_A \cong k$, as above, the group

$$\mathscr{T}_X^{i,n}(A) = \mathrm{H}^i(X, \mathcal{K}^M_{n,\rho}).$$

Our goal in this section is to prove the following theorem.

Theorem 3.1. *Suppose that X is a smooth, proper, and geometrically connected k-variety. Fix an integer $j \geq 1$, and suppose also that the following conditions are satisfied:*

(1) $\mathrm{H}^j(X, \mathcal{O}_X) = \mathrm{H}^{j+1}(X, \mathcal{O}_X) = \mathrm{H}^{j+2}(X, \mathcal{O}_X) = 0$,
(2) $\mathrm{H}^j(X, \Omega^1_{X/k}) = \mathrm{H}^{j+1}(X, \Omega^1_{X/k}) = 0$.

Then there is a canonical isomorphism

$$\mathscr{T}_X^{j,3}(A) = \mathrm{H}^j(X, \Omega^2_{X/k}) \otimes_k \mathfrak{m}_A$$

for any Artinian local k-algebra $(A, \mathfrak{m}_A)$ as above. In particular, this implies that the functor $\mathscr{T}_X^{j,3}$ is pro-representable.

Example 3.2. The assumptions of Theorem 3.1 are satisfied if $j = 3$ and X is a smooth complete intersection of two quadrics in $\mathbb{P}^7$ or if X is a smooth cubic hypersurface in $\mathbb{P}^6$, see [Rap72, §2]. If $j = 3$ still, the assumptions are also satisfied when X is a Gushel-Mukai fivefold [DK19, Proposition 3.1].

The proof of Theorem 3.1 relies crucially on the next two lemmas.

Lemma 3.3. *Suppose that X is geometrically integral. Fix some $j \geq 1$. Assume also that:*

(1) $\mathrm{H}^j(X, \mathcal{O}_X) = \mathrm{H}^{j+1}(X, \mathcal{O}_X) = 0$,
(2) and $\mathrm{H}^j(X, \Omega^1_{X/k}) = \mathrm{H}^{j+1}(X, \Omega^1_{X/k}) = 0$.

Then we have $\mathrm{H}^j(X, \mathcal{C}^2_\rho) = 0$, and hence also

$$\mathrm{H}^j(X, \mathcal{R}^2_\rho/d\mathcal{R}^1_\rho) = \mathrm{H}^j(X, \Omega^2_{X/k} \boxtimes \mathfrak{m}_A / d\Omega^1_{X/k} \otimes_k \mathfrak{m}^0_A),$$

for any Artinian local k-algebra $(A, \mathfrak{m}_A)$ with $A/\mathfrak{m}_A \cong k$ as above.

Proof. To prove the lemma we'll use the exact sequence

$$0 \longrightarrow \left(\Omega^1_{X/k} \otimes_k d(\mathfrak{m}_A)\right) \cap d(\mathcal{O}_X \boxtimes \Omega^1_{A/k}) \to \left(\Omega^1_{X/k} \otimes_k d(\mathfrak{m}_A)\right) \oplus d(\mathcal{O}_X \boxtimes \Omega^1_{A/k}) \to \left(\Omega^1_{X/k} \boxtimes \Omega^1_{A/k}\right) \oplus \left(\mathcal{O}_X \boxtimes \Omega^2_{A/k}\right) \longrightarrow \mathcal{C}^2_\rho \longrightarrow 0$$

where the second arrow is $(+,-)$ and the third is the sum $(a,b)\mapsto a+b$. We're going to identify the cohomology groups of each of these sheaves; then we'll patch together some long-exact cohomology sequences and deduce the result.

To simplify our notation, we write

$$D=\left(\Omega^1_{X/k}\otimes_k d(\mathfrak{m}_A)\right)\cap d(\mathcal{O}_X\boxtimes\Omega^1_{A/k})$$

and we set K to be the kernel of the third arrow. By splicing the four-term sequence above we get two short exact sequences

$$0\to D\to\left(\Omega^1_{X/k}\otimes_k d(\mathfrak{m}_A)\right)\oplus d(\mathcal{O}_X\boxtimes\Omega^1_{A/k})\to K\to 0 \tag{S1}$$

and

$$0\to K\to\left(\Omega^1_{X/k}\boxtimes\Omega^1_{A/k}\right)\oplus\left(\mathcal{O}_X\boxtimes\Omega^2_{A/k}\right)\to\mathcal{C}^2_\rho\to 0. \tag{S2}$$

The first nontrivial term D can be identified as

$$D=\left(\Omega^1_{X/k}\otimes_k d(\mathfrak{m}_A)\right)\cap d(\mathcal{O}_X\boxtimes\Omega^1_{A/k})=d(\mathcal{O}_X)\otimes_k d(\mathfrak{m}_A)\subset\Omega^1_{X/k}\boxtimes\Omega^1_{A/k}.$$

By [Har77, Remark 2.9.1] sheaf cohomology commutes with arbitrary (especially finite) direct sums, so this yields an isomorphism

$$\begin{aligned}\mathrm{H}^i(X,D)&=\mathrm{H}^i(X,d(\mathcal{O}_X)\otimes_k d(\mathfrak{m}_A))\\&\cong\mathrm{H}^i(X,d(\mathcal{O}_X))\otimes_k d(\mathfrak{m}_A).\end{aligned}$$

Since X is geometrically integral, hence irreducible, the constant sheaf $\underline{k}$ is flasque. From the long exact cohomology sequence associated to the short exact sequence

$$0\to\underline{k}\to\mathcal{O}_X\xrightarrow{d}d(\mathcal{O}_X)\to 0$$

this, in turn, implies that, for any $i\geq 1$, there is an isomorphism

$$\mathrm{H}^i(X,d(\mathcal{O}_X))\cong\mathrm{H}^i(X,\mathcal{O}_X)$$

since the higher cohomology of a flasque sheaf vanishes by [Har77, Proposition 2.5]. Altogether, this produces an isomorphism

$$\mathrm{H}^i(X,D)\cong\mathrm{H}^i(X,\mathcal{O}_X)\otimes_k d(\mathfrak{m}_A)$$

for any $i\geq 1$.

To compute the cohomology of the middle term in the sequence (S1), we need to write $d(\mathcal{O}_X\boxtimes\Omega^1_{A/k})$ in a way that allows us to compute its cohomology. But note that the differential

$$\mathcal{O}_X\boxtimes\Omega^1_{A/k}\xrightarrow{d}\left(\Omega^1_{X/k}\boxtimes\Omega^1_{A/k}\right)\oplus\left(\mathcal{O}_X\boxtimes\Omega^2_{A/k}\right)$$

has kernel $\underline{k}\otimes_k L$ where $L = \ker(\Omega^1_{A/k} \to \Omega^2_{A/k})$. By an argument similar to before, noting that $\underline{k} \otimes_k L$ is also flasque, we have isomorphisms for all $i > 0$

$$\begin{aligned} \mathrm{H}^i(X, d(\mathcal{O}_X \boxtimes \Omega^1_{A/k})) &\cong \mathrm{H}^i(X, \mathcal{O}_X \boxtimes \Omega^1_{A/k}) \\ &\cong \mathrm{H}^i(X, \mathcal{O}_X) \otimes_k \Omega^1_{A/k}. \end{aligned}$$

The long-exact cohomology sequence associated to (S1) now breaks up into short exact sequences, for every $i \geq 1$,

$$0 \to \mathrm{H}^i(X, \mathcal{O}_X) \otimes d(\mathfrak{m}_A) \to \left(\mathrm{H}^i(X, \Omega^1_{X/k}) \otimes d(\mathfrak{m}_A)\right) \oplus \left(\mathrm{H}^i(X, \mathcal{O}_X) \otimes \Omega^1_{A/k}\right) \\ \to \mathrm{H}^i(X, K) \to 0.$$

Our assumptions on the vanishing of the cohomology of $\mathcal{O}_X$ and $\Omega^1_{X/k}$ imply that both $\mathrm{H}^j(X, K)$ and $\mathrm{H}^{j+1}(X, K)$ vanish.

The long-exact sequence associated to (S2) now shows that

$$\mathrm{H}^j(X, \mathcal{C}^2_\rho) \cong \left(\mathrm{H}^j(X, \Omega^1_{X/k}) \otimes \Omega^1_{A/k}\right) \oplus \left(\mathrm{H}^j(X, \mathcal{O}_X) \otimes \Omega^2_{A/k}\right).$$

But both summands of this latter space vanish by assumption. Lastly, the claim that

$$\mathrm{H}^i(X, \mathcal{R}^2_\rho / d\mathcal{R}^1_\rho) = \mathrm{H}^i(X, \left(\Omega^2_{X/k} \boxtimes \mathfrak{m}_A\right) / \left(d\Omega^1_{X/k} \otimes_k \mathfrak{m}^0_A\right))$$

follows as a consequence of Lemma 2.6. □

Remark 3.4. If X is a smooth, proper, and geometrically integral k-variety, then the map $\mathrm{H}^*(X, \mathcal{O}_X) \to \mathrm{H}^*(X, \Omega^1_{X/k})$ induced by the differential vanishes (this follows from the degeneration of the Hodge to de Rham spectral sequence proved, for example, in [DI87, Corollaire 2.7]). In this case, the above proof can be modified (with no assumptions on the vanishing of the cohomology of either $\mathcal{O}_X$ or $\Omega^1_{X/k}$) to show that

$$\mathrm{H}^i(X, \mathcal{C}^2_\rho) \cong \left(\mathrm{H}^i(X, \Omega^1_{X/k}) \otimes_k \Omega^1_{A/k}/d(\mathfrak{m}_A)\right) \oplus \left(\mathrm{H}^i(X, \mathcal{O}_X) \otimes_k \Omega^2_{A/k}/d(\Omega^1_{A/k})\right)$$

for any $i \geq 1$. Hence, if for some fixed $j \geq 1$ one has $\mathrm{H}^j(X, \Omega^1_{X/k}) = 0$ and $\mathrm{H}^j(X, \mathcal{O}_X) = 0$, then $\mathrm{H}^j(X, \mathcal{C}^2_\rho) = 0$.

Lemma 3.5. *Suppose that X is a smooth, proper, and geometrically connected k-variety. Fix two integers $p \geq j \geq 1$, and suppose also that the following conditions are satisfied:*

(1) $\mathrm{H}^{p+q}(X, \Omega^{j-q-1}_{X/k}) = 0$ *for all* $0 \leq q \leq j - 1$

(2) and $\mathrm{H}^{p+q}(X, \Omega^{j-q}_{X/k}) = 0$ *for all* $1 \leq q \leq j$

Then the canonical quotient map induces an isomorphism

$$\mathrm{H}^p(X, \Omega^j_{X/k}) \otimes_k \mathfrak{m}_A \cong \mathrm{H}^p\left(X, \frac{\Omega^j_{X/k} \boxtimes \mathfrak{m}_A}{d\Omega^{j-1}_{X/k} \otimes_k \mathfrak{m}^0_A}\right)$$

for any Artinian local k-algebra $(A, \mathfrak{m}_A)$ *with* $A/\mathfrak{m}_A \cong k$ *as above.*

Proof. We write

$$\mathscr{K}^i = \ker\left(\Omega^i_{X/k} \xrightarrow{d^i} \Omega^{i+1}_{X/k}\right)$$

for the given sheaf kernel and

$$\mathscr{H}^i = \mathrm{H}\left(\Omega^{i-1}_{X/k} \to \Omega^i_{X/k} \to \Omega^{i+1}_{X/k}\right)$$

for the given homology sheaf, i.e. the quotient of $\mathscr{K}^i$ by the image sheaf of the first morphism. We don't use it but, the sheaf $\mathscr{H}^i$ can also be identified with the sheafification of the presheaf associating to an open $U \subset X$ the algebraic de Rham cohomology $\mathrm{H}^i_{dR}(U)$.

Using this notation, there are the following exact sequences:

$$(D_0) \qquad 0 \to \underline{k} \to \mathcal{O}_X \to d\mathcal{O}_X \to 0,$$

$$(H_i) \qquad 0 \to d\Omega^{i-1}_{X/k} \to \mathscr{K}^i \to \mathscr{H}^i \to 0 \quad \text{for } i \geq 1,$$

$$(D_i) \qquad 0 \to \mathscr{K}^i \to \Omega^i_{X/k} \to d\Omega^i_{X/k} \to 0 \quad \text{for } 1 \leq i \leq j-1,$$

$$(D_j) \qquad 0 \to d\Omega^{j-1}_{X/k} \otimes_k \mathfrak{m}^0_A \to \Omega^j_{X/k} \boxtimes \mathfrak{m}_A \to \frac{\Omega^j_{X/k} \boxtimes \mathfrak{m}_A}{d\Omega^{j-1}_{X/k} \otimes_k \mathfrak{m}^0_A} \to 0.$$

The long exact sequence associated to (D_j) shows that, in order to prove the lemma, it suffices to show the simultaneous vanishing

$$\mathrm{H}^p(X, d\Omega^{j-1}_{X/k}) = \mathrm{H}^{p+1}(X, d\Omega^{j-1}_{X/k}) = 0.$$

Looking at the sequence (D_i) when $i = j-1$, this is implied by the simultaneous vanishing

$$\mathrm{H}^p(X, \Omega^{j-1}_{X/k}) = \mathrm{H}^{p+1}(X, \mathscr{K}^{j-1}) = 0$$
$$\text{and } \mathrm{H}^{p+1}(X, \Omega^{j-1}_{X/k}) = \mathrm{H}^{p+2}(X, \mathscr{K}^{j-1}) = 0.$$

The cohomology of $\Omega^{j-1}_{X/k}$ vanishes by assumption and, from the long exact sequence of cohomology associated with (H_i) when $i = j-1$, the vanishing of the cohomology of $\mathscr{K}^{j-1}$ is implied by the simultaneous vanishing

$$\mathrm{H}^{p+1}(X, d\Omega^{j-2}_{X/k}) = \mathrm{H}^{p+1}(X, \mathscr{H}^{j-1}) = 0$$
$$\text{and } \mathrm{H}^{p+2}(X, d\Omega^{j-2}_{X/k}) = \mathrm{H}^{p+2}(X, \mathscr{H}^{j-1}) = 0.$$

According to [BO74, Corollary 6.2], the cohomology $\mathrm{H}^a(X, \mathscr{H}^b) = 0$ vanishes whenever $a > b$ (which is the case when $a = p$ and $b = j-1$ by assumption). The claim then follows from repeating this argument, eventually reducing to a computation of the terms of the long exact sequence associated to (D_0). □

Proof of Theorem 3.1. The isomorphism

$$\mathscr{T}_X^{j,3}(A) \cong \mathrm{H}^j(X, \Omega^2_{X/k}) \otimes_k \mathfrak{m}_A$$

follows immediately from Lemmas 2.4, 2.6, 3.3, and 3.5.

That $\mathscr{T}_X^{j,3}$ is pro-representable can be checked using Schlessinger's Criterion [Sch68, Theorem 2.11]. We'll show, instead, that, under the assumptions of the theorem statement, the functor $\mathscr{T}_X^{j,3}$ has a tangent-obstruction theory; the sufficiency of this condition to guarantee pro-representability is proved in [FG05, Corollary 6.3.5]. Specifically, we'll show that there exist finite dimensional k-vector spaces T_1, T_2 and an exact sequence

$$0 \to T_1 \otimes_k J \to \mathscr{T}_X^{j,3}(A') \to \mathscr{T}_X^{j,3}(A) \to T_2 \otimes_k J \tag{T-O}$$

whenever there exists a small-extension

$$0 \to J \to A' \to A \to 0$$

of local Artinian k-algebras $(A, \mathfrak{m}_A)$ and $(A', \mathfrak{m}_{A'})$ with residue fields isomorphic to k (recall that A' is a small extension of A if $J \cdot \mathfrak{m}_{A'} = 0$). It will be clear from the construction that the exact sequence (T-O) is functorial in small-extensions, and this will complete the proof.

So, assume we're in the setting above with A, A', J given. We'll see that $T_1 \cong \mathrm{H}^j(X, \Omega^2_{X/k})$ and $T_2 = 0$. From the surjection $A' \to A$ we get a commutative ladder with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{T}_X^{j,3}(A') & \longrightarrow & \mathrm{H}^i(X_{A'}, \mathcal{K}^M_{3,X_{A'}}) & \longrightarrow & \mathrm{H}^i(X, \mathcal{K}^M_{3,X}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathscr{T}_X^{j,3}(A) & \longrightarrow & \mathrm{H}^i(X_A, \mathcal{K}^M_{3,X_A}) & \longrightarrow & \mathrm{H}^i(X, \mathcal{K}^M_{3,X}) & \longrightarrow & 0
\end{array}$$

In view of the isomorphisms

$$\mathscr{T}_X^{j,3}(A') \cong \mathrm{H}^j(X, \Omega^2_{X/k}) \otimes_k \mathfrak{m}_{A'} \quad \text{and} \quad \mathscr{T}_X^{j,3}(A) \cong \mathrm{H}^j(X, \Omega^2_{X/k}) \otimes_k \mathfrak{m}_A,$$

the leftmost vertical arrow is a surjection with kernel $\mathrm{H}^j(X, \Omega^2_{X/k}) \otimes_k J$. This immediately proves the theorem, by the comments above. □

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Email address: eoinmackall *at* gmail.com
URL: www.eoinmackall.com