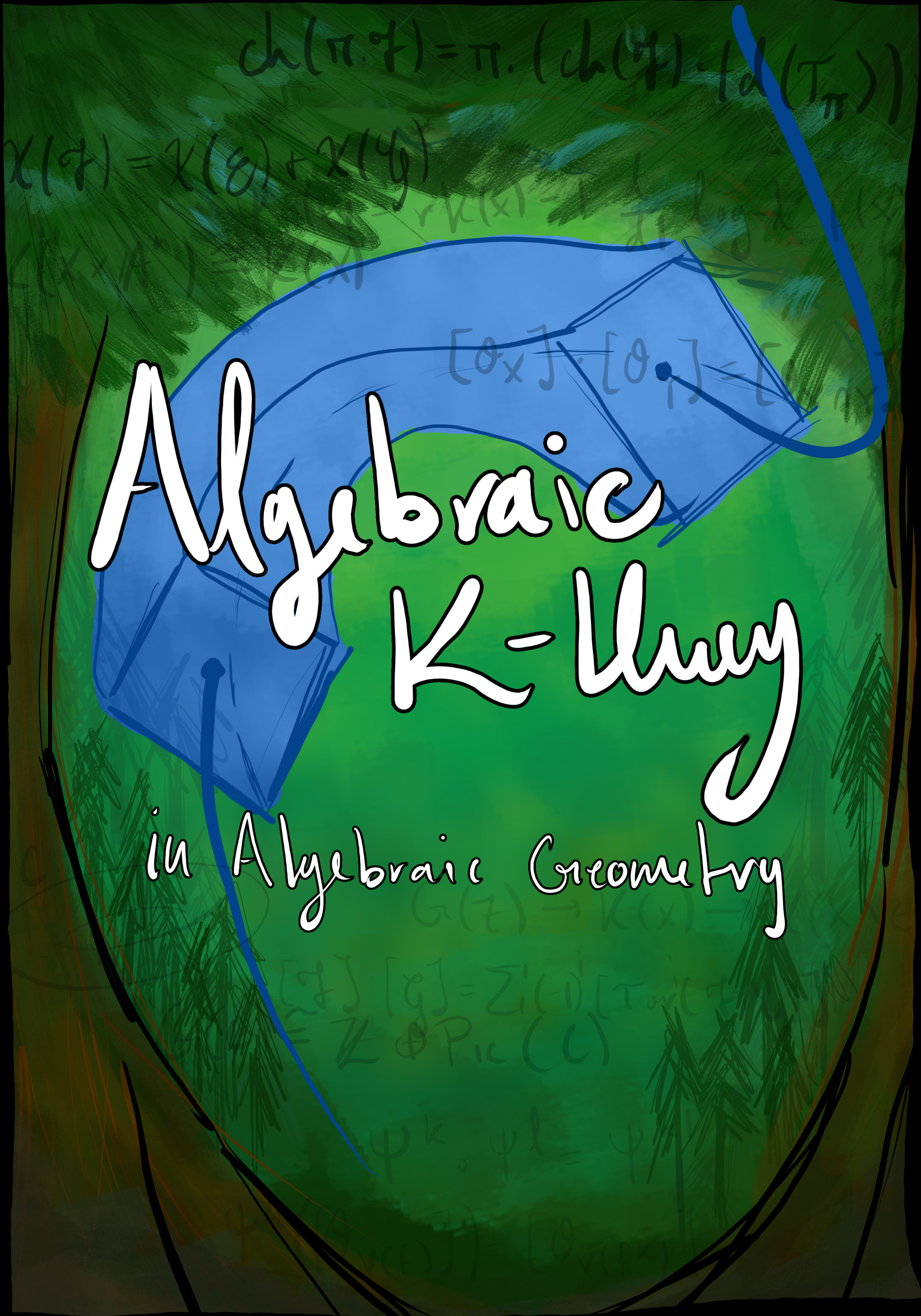
Algebraic
K-Theory
in Algebraic Geometry

Algebraic K-theory in Algebraic Geometry

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October 27, 2022

This book is meant to be an exposition on Grothendieck's K-functor and it's role in algebraic geometry. It is not supposed to be a treatise. As such, I've chosen to exclude mention of higher K-theory, derived categories, and $\mathbb{A}^1$-homotopy theory. I hope that, by doing so, I've made this book more approachable to an interested algebraic geometer. This book is free to download; it can be found on the author's webpage: https://www.eoinmackall.com/.

The book is written with the early-career algebraic-geometer in mind. Ideally, the audience who will find the book most useful will be those with a solid background in commutative algebra, who have had some introduction to both scheme theory and the derived functor language.

References are ordered lexicographically in the format (Chapter.Section.Reference). Exercises are given at the end of each section. These vary in difficulty and exercises that are marked with an asterisk indicate that they rely on material or information outside of what is assumed throughout this book.

Contents

Conventions

We assume that every ring is unital. Often we denote the identity element of a ring R by 1_R or just by 1 when the ring R is clear from context. We have that $1_R = 0_R$ if and only if $R = 0$. All homomorphisms $\varphi : R \to S$ between commutative rings R and S satisfy $\varphi(1_R) = 1_S$.

The empty scheme is denoted by $\emptyset$. By definition, this is the pair $(\emptyset, \mathcal{O}_\emptyset)$ whose underlying set is the empty set $\emptyset$ and whose structure sheaf $\mathcal{O}_\emptyset$ is characterized by the assignment $\mathcal{O}_\emptyset(\emptyset) = 0$.

Throughout this book we reserve the letter k for an arbitrary but fixed field. Given a field extension F/k we will say that X is an F-variety if X is a separated scheme of finite type over F. When the field F is clear from context, we'll simply say that X is a variety.

Introduction

K-Theory and G-theory of Rings

In this chapter we begin our study of K-theory and G-theory in the affine setting. There are two reasons for treating this material first. For one, it's useful to have examples where most of the theory can be worked out with minimal prerequisites. For the other, we'll prove some theorems here (in the affine case) that will form the bases for arguments in later chapters (for the general case).

The chapter starts by recalling the notion of a projective module in Section 1.1. These objects are foundational to algebraic K-theory and will serve as the main actors in this text. Here we give two characterizations of a projective module: we first show that a module is finitely generated and projective if and only if it is a direct summand of a finite rank free module if and only if it is flat and finitely presented (Theorem 1.1.10); we next show that finitely generated and projective modules are precisely those modules which are locally free of finite rank on the corresponding affine scheme (Theorem 1.1.13).

The K-theory ring $K(R)$, associated to a ring R, is introduced in Section 1.2. We comment on some of the basic properties of K-theory, but we leave a thorough treatment of its intricate functorial properties to later chapters. In this section we focus mainly on computations. In order to convince the reader that the K-theory of a ring is a highly nontrivial object, we also introduce in this section the (possibly more-familiar) Picard group of a ring as the group which classifies all invertible modules for the ring (i.e. finitely generated and projective modules of rank one). We then show how the Picard group of an integral domain R can be realized as a quotient of $K(R)$ and we use this observation to give some examples of rings R where $K(R)$ is very much nontrivial.

Section 1.3 serves as a bridge connecting the K-theory from the previous section to the G-theory of a ring introduced in the following Section 1.4. This section, which focuses on divisors, is one of the more technical sections in this chapter. When deciding how to handle the material, I eventually settled on the opinion that providing a very thorough treatment of this theory in the case of rings could serve as motivation to the more abstract treatment that's usually presented for schemes. Both fractional ideals and Weil divisors are introduced in this section and their connections to the Picard group and the divisor class group are respectively explained. Notable theorems of this section include proving that the Picard group of an integrally closed domain includes into the divisor class group with equality if and only if the integrally closed domain is locally factorial (Theorem 1.3.27) and proving that the divisor class group of an integrally closed domain vanishes if and only if that ring is a unique factorization domain (Corollary 1.3.33).

In Section 1.4, we introduce the G-theory group $G(R)$ associated to a ring R. The group $G(R)$ is more intimately connected to geometry, and we try to illustrate this by providing a lesser-known description of $G(R)$ in terms of algebraic cycles. This description also allows us to show that for an integrally closed domain R, the group $G(R)$ has, as a canonical subquotient, the divisor class group of R introduced in the previous section. In some ways, even though the title of this book references only K-theory, it's because of these relations to more classical objects in geometry that the group $G(R)$ is the truly interesting object from the point of view of an algebraic geometer.

Section 1.5 is effectively the last section of this chapter which is noticeably related to our aim of studying algebraic geometry. Here we introduce regular rings and study their homological and algebraic properties. Two monumentous theorems are proved in this section: the Auslander-Buchsbaum theorem (Theorem 1.5.12) which characterizes regular local rings as exactly those rings with the property that every finitely generated module has finite projective dimension, and an analog of Poincaré duality (Theorem 1.5.3) showing that $K(R)$ and $G(R)$ are isomorphic for a regular ring R of finite Krull dimension. Still, the content of Section 1.5 differs quite a bit in nature from that of the previous sections. Some of this difference is accounted for in the fact that this is the only section in which a majority of results rely on arguments based in homological algebra. However, even the pacing of this section is at a different level than most of material in the rest of this chapter (maybe one can view this as a necessary trade-off relative to the importance of the results obtained).

Finally, in Section 1.6, we recall the basic structure theorems of semisimple algebras over a field and we define the K-theory of a noncommutative algebra. Although this material may seem temporarily out of place for those interested in studying geometry, these objects will turn out to play an important role in describing the K-theory of certain varieties such as the quadrics and Severi–Brauer varieties of later chapters.

I've had access to an invaluable wealth of resources while writing this chapter. Some of the more standard references which have had a direct impact on this text include [Wei13], which was very helpful for organizing my thoughts on the material in Section 1.3 and from which many examples throughout the text originate, and [Ros94], which is where I first learned most of the proofs here on the K-theory of Dedekind domains.

Aside from these standard references, the presentation of this chapter has also been heavily influenced by the lecture notes of Mel Hochster (available online). In particular, this is where I learned both the proof of Theorem 1.4.19, which Hochster attributes to M. P. Murthy, and the proof of the Auslander-Buchsbaum theorem as it's presented here.

1.1 Projective modules

Let R be a commutative ring.

Definition 1.1.1. An R-module M is said to be *projective* if for any pair of R-modules N and L, and for any diagram

$$\begin{tikzcd} & M \arrow[d] \arrow[dl, dotted] & \\ L \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

with the bottom row exact, there exists a morphism $M \to L$ that fills the dotted arrow to make a commuting triangle.

Remark 1.1.2. Equivalently, an R-module M is projective if and only if the functor $\mathrm{Hom}_R(M, -)$ is right-exact; see Exercise 1.1.1.

There are a number of reasons to be concerned with projective modules (and, as a word towards the prerequisites for this book, hopefully the reader has already encountered some of these reasons!). For instance, projective modules can be used to construct both the $\mathrm{Tor}_*^R(-,-)$ and $\mathrm{Ext}_R^*(-,-)$ functors of homological algebra. We will have ample reason to use these functors and their properties throughout this section (even more-so throughout the rest of the book) and we often will use them without much explanation.

It's largely because of their role in homological algebra that projective modules also appear as central characters in K-theory. However, in contrast to the situation in homological algebra, we won't be interested in arbitrary projective modules; we'll only be interested in those modules satisfying either of the following finiteness conditions.

Definition 1.1.3. An R-module M is said to be *finitely generated* if there is a short exact sequence of R-modules

$$R^{\oplus n} \to M \to 0$$

for some nonnegative integer $n \in \mathbb{Z}_{\geq 0}$.

Definition 1.1.4. An R-module M is said to be *finitely presented* if there is a short exact sequence of R-modules

$$R^{\oplus m} \to R^{\oplus n} \to M \to 0$$

for some nonnegative integers $m, n \in \mathbb{Z}_{\geq 0}$.

Example 1.1.5. The free R-module $R^{\oplus I}$ on an indexing set I is always projective. It's finitely presented if and only if the cardinality $\#I$ of I is finite, i.e. $\#I \in \mathbb{Z}_{\geq 0}$. In the case that $\#I$ is finite, we say that $R^{\oplus I}$ is free of *rank* $\#I$ and we write $\mathrm{rk}_R(R^{\oplus I}) = \#I$ or simply $\mathrm{rk}(R^{\oplus I}) = \#I$ if no confusion will occur.

Remark 1.1.6. Any finitely presented R-module is necessarily finitely generated. If R is Noetherian then the converse, that every finitely generated R-module M is finitely presented, is also true.

Our goal for this section is to set-up basic results on projective modules which will be used, often implicitly, throughout the remainder of this book. There are two main theorems proved here. The first, Theorem 1.1.10, gives strong algebraic restrictions on the class of projective modules. Namely, Theorem 1.1.10 says that a module is finitely generated and projective if and only if it is finitely presented and flat if and only if it is a direct summand of a free module of finite rank.

The second main theorem of this section, Theorem 1.1.13, characterizes finitely generated and projective modules geometrically as precisely those modules that are locally free of finite rank. This observation, first made by Serre in a geometric setting [Ser55] and proved in greater generality later by Kaplanksy [Kap58], is key to setting-up a correspondence between finitely generated projective modules over a ring and vector bundles on the associated affine scheme.

As a warm-up for the proofs of these theorems, and because we will also need the statements, we explore some relations between the concepts just introduced.

Lemma 1.1.7. *Let $R \neq 0$ be a commutative ring. Let L, M, and N be R-modules. Then the following hold.*

(1) *If there is a short exact sequence*

$$M \to N \to 0$$

with M finitely generated, then N is finitely generated.

(2) *If there is a short exact sequence*

$$0 \to L \to M \to N \to 0$$

with N finitely presented and with M finitely generated, then L is finitely generated.

Proof. For (1), a surjection $R^{\oplus n} \to M$ gives a surjection $R^{\oplus n} \to M \to N$. For (2), one can extend a finite presentation of N to a commutative ladder (i.e. so that all squares commute) with exact rows

$$\begin{tikzcd} & R^{\oplus m} \arrow[r] \arrow[d] & R^{\oplus n} \arrow[r] \arrow[d] & N \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & L \arrow[r] & M \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

using the projectivity of free modules. The snake lemma then shows that

$$\operatorname{coker}(R^{\oplus m} \to L) = \operatorname{coker}(R^{\oplus n} \to M).$$

Since M is finitely generated, the cokernel on the right-hand-side of the equality above is finitely generated by (1). Thus L is finitely generated as well (a generating set for L is given by the choice of a finite generating set for the image of $R^{\oplus m}$ in L and a choice of lifts of a finite generating set for $\operatorname{coker}(R^{\oplus m} \to L)$). $\square$

Lemma 1.1.8. *Let $R \neq 0$ be a local ring with maximal ideal $\mathfrak{m} \subset R$ and let M be an R-module. If M is flat and finitely presented, then M is free with finite rank.*

Proof. Since M is finitely generated, the quotient $M/\mathfrak{m}M$ is a finite dimensional $R/\mathfrak{m}$-vector space. Let $e_1, ..., e_n$ be elements of M that form a basis for $M/\mathfrak{m}M$. By Nakayama's lemma [AM69, Proposition 2.8], the elements $e_1, ..., e_n$ generate M so that there is a short exact sequence induced by sending the ith standard basis element of $R^{\oplus n}$ to e_i,

$$0 \to K \to R^{\oplus n} \to M \to 0$$

and with K the appropriate kernel. Since M is finitely presented, Lemma 1.1.7 (2) shows that K is finitely generated. Since M is a flat R-module, $\operatorname{Tor}_1^R(M, R/\mathfrak{m}) = 0$ and tensoring by $R/\mathfrak{m}$ yields an exact sequence

$$0 \to K \otimes_R R/\mathfrak{m} \to (R \otimes_R R/\mathfrak{m})^{\oplus n} \to M \otimes_R R/\mathfrak{m} \to 0.$$

The leftmost $R/\mathfrak{m}$-vector space $K \otimes_R R/\mathfrak{m} = K/\mathfrak{m}K = 0$ vanishes by construction. Applying Nakayama's lemma (again) shows that $K = 0$; hence $M = R^{\oplus n}$. $\square$

Lemma 1.1.9. *Let $R \neq 0$ be a given ring and let $R \to S$ be a ring extension. Then for any R-modules M and N there is a natural homomorphism*

$$\operatorname{Hom}_R(M, N) \otimes_R S \to \operatorname{Hom}_S(M \otimes_R S, N \otimes_R S).$$

If S is flat over R, and if M is finitely presented, then this homomorphism is an isomorphism.

Proof. To define the natural map, we can use the universal property of tensor products. The association

$$\operatorname{Hom}_R(M, N) \times S \to \operatorname{Hom}_S(M \otimes_R S, N \otimes_R S) \qquad (f, s) \mapsto s \cdot f$$

is well-defined and R-bilinear, hence descends to a map from the tensor product. Assume then that S is flat and M is finitely presented with presentation

$$R^{\oplus m} \to R^{\oplus n} \to M \to 0.$$

This presentation gives rise to the following commutative ladder with exact rows.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}_R(M,N)\otimes_R S & \longrightarrow & (N\otimes_R S)^{\oplus n} & \longrightarrow & (N\otimes_R S)^{\oplus m} \\ & & \big\downarrow{\scriptstyle\alpha} & & \| & & \| \\ 0 & \longrightarrow & \mathrm{Hom}_S(M\otimes_R S, N\otimes_R S) & \longrightarrow & (N\otimes_R S)^{\oplus n} & \longrightarrow & (N\otimes_R S)^{\oplus m} \end{array}$$

Here the top row comes from the presentation for M by applying the functor $\mathrm{Hom}_R(-,N)$ and then the functor $-\otimes_R S$; the bottom row comes from applying the functor $-\otimes_R S$ and then $\mathrm{Hom}_S(-,N\otimes_R S)$. The arrow labeled α exists by a diagram chase which, if one traces through all the isomorphisms, agrees with the natural homomorphism considered above. Lastly, to see that α is an isomorphism one can apply the five lemma. □

Our first structure theorem for projective modules is the following:

Theorem 1.1.10. *Let $R \neq 0$ be an arbitrary ring and let M be an R-module. Then the following conditions on M are equivalent:*
(1) *M is finitely generated and projective,*
(2) *M is a direct summand of a free R-module $R^{\oplus n}$ for some $n \in \mathbb{N}$,*
(3) *M is finitely presented and flat as an R-module.*

Proof. We're going to show that (1) $\implies$ (2) $\implies$ (3) $\implies$ (1). So assume (1). Since M is finitely generated there is an exact sequence

$$R^{\oplus n} \to M \to 0$$

with $n \in \mathbb{N}$. Since M is projective, this sequence splits (take $M = N$ with the identity map in Definition 1.1.1). Hence (2) holds.

Now we show (2) $\implies$ (3). Let P be an R-module so that $M \oplus P = R^{\oplus n}$. Tensoring any short exact sequence of R-modules

$$0 \to A \to B \to C \to 0,$$

by $M \oplus P$ gives a commuting ladder like the following.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^{\oplus n} & \longrightarrow & B^{\oplus n} & \longrightarrow & C^{\oplus n} & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ 0 & \longrightarrow & A\otimes M \oplus A\otimes P & \longrightarrow & B\otimes M\oplus B\otimes P & \longrightarrow & C\otimes M \oplus C\otimes P & \longrightarrow & 0 \end{array}$$

Homology of a complex commutes with direct sums of complexes. Hence the top row is exact, since it is a sum of short exact sequences, and this implies the bottom is exact as well. Using the commutativity of homology and sums once again, it follows that the sequence

$$0 \to A \otimes M \to B \otimes M \to C \otimes M \to 0$$

is exact, proving that M is flat.

To see that M is finitely presented, we note that as summands of $R^{\oplus n}$ both M and P are finitely generated (apply Lemma 1.1.7 (1) to the projections from $M \oplus P$). Concatenating the associated maps

$$\begin{tikzcd} R^{\oplus m} \arrow[r] \arrow[dr, two heads] & R^{\oplus n} \arrow[r] & M \arrow[r] & 0 \\ & P \arrow[u, hook] & & \end{tikzcd}$$

gives a finite presentation for M.

Lastly, we show (3) $\implies$ (1). To do this, we show that the functor $\mathrm{Hom}_R(M, -)$ is right-exact (see Remark 1.1.2 and Exercise 1.1.1). So let

$$B \to C \to 0$$

be an exact sequence of R-modules. For each prime ideal $\mathfrak{p} \subset R$, the localized

$$B_\mathfrak{p} \to C_\mathfrak{p} \to 0$$

is exact as a sequence of $R_\mathfrak{p}$-modules. Since M is flat and finitely presented, the localization $M_\mathfrak{p} = R_\mathfrak{p}^{\oplus n}$ is a free $R_\mathfrak{p}$-module by Lemma 1.1.8. It follows that there is a commutative diagram

$$\begin{tikzcd} B_\mathfrak{p}^{\oplus n} \arrow[r] \arrow[d, equal] & C_\mathfrak{p}^{\oplus n} \arrow[r] \arrow[d, equal] & 0 \\ \mathrm{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p}, B_\mathfrak{p}) \arrow[r] & \mathrm{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p}, C_\mathfrak{p}) \arrow[r] & 0 \end{tikzcd}$$

with exact rows for each prime ideal $\mathfrak{p} \subset R$. But, because of Lemma 1.1.9, this means that the sequence

$$\mathrm{Hom}_R(M, B) \to \mathrm{Hom}_R(M, C) \to 0$$

is exact when localized at each prime $\mathfrak{p} \subset R$. Since surjectivity can be checked locally [AM69, Proposition 3.9], this completes the proof. $\square$

In the remainder of this section we characterize finitely generated projective R-modules as exactly those R-modules whose associated quasicoherent sheaves are locally free of finite rank on $\mathrm{Spec}(R)$.

Lemma 1.1.11. *Let $R \neq 0$ be a commutative ring and let M be an R-module. Then the following conditions on M are equivalent:*

(1) *M is finitely presented and for every prime ideal $\mathfrak{p} \subset R$, the localization $M_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module.*
(2) *M is finitely presented and for every maximal ideal $\mathfrak{m} \subset R$, the localization $M_{\mathfrak{m}}$ is a free $R_{\mathfrak{m}}$-module.*
(3) *M is finitely generated and there exist elements $f_i \in R$ such that the ideal generated by the f_i is all of R, i.e. $\sum_i (f_i) = R$, and all of the localizations M_{f_i} are free R_{f_i}-modules.*
(4) *There exist elements $f_i \in R$ such that $\sum_i (f_i) = R$, and the localizations M_{f_i} are free over R_{f_i} of finite rank.*
(5) *M is finitely generated, for every prime ideal $\mathfrak{p} \subset R$ the localization $M_{\mathfrak{p}}$ is free, and the assignment*

$$\mathrm{rk}_R(M, -) : \mathrm{Spec}(R) \to \mathbb{Z} \quad \textit{where} \quad \mathrm{rk}_R(M, \mathfrak{p}) := \mathrm{rk}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$$

is locally constant for the Zariski topology on $\mathrm{Spec}(R)$.

Proof. The implications (1) $\implies$ (2) and (3) $\implies$ (4) are immediate. We'll show the remaining implications (2) $\implies$ (3) and (4) $\implies$ (5) $\implies$ (1).

Assume (2). For every maximal ideal $\mathfrak{m} \subset R$ we can find an isomorphism

$$R_{\mathfrak{m}}^{\oplus r} \xrightarrow{\sim} M_{\mathfrak{m}}$$

where $r = \mathrm{rk}(M, \mathfrak{m})$. Label the standard basis of the free module on the left hand side of this isomorphism $e_1, ..., e_r$ and label $f_1, ..., f_r$ their images in $M_{\mathfrak{m}}$. Clearing denominators by an element $g \in R \setminus \mathfrak{m}$ if necessary, one can assume that $f_1, ..., f_r$ are the images of elements $\tilde{f}_1, ..., \tilde{f}_r$ from the localization $M \to M_{\mathfrak{m}}$. The map

$$\alpha : R^{\oplus r} \to M$$

defined by sending $e_1, ..., e_r$ to $\tilde{f}_1, ..., \tilde{f}_r$ is an isomorphism after localizing by $\mathfrak{m}$. Since M is finitely generated, the cokernel of α is also finitely generated by Lemma 1.1.7 (1). We can then pick an element $g \in R \setminus \mathfrak{m}$ that annihilates all of the generators of $\mathrm{coker}(\alpha)$ simultaneously. This means that the localization

$$\alpha_g : R_g^{\oplus r} \to M_g$$

is surjective and, since M is finitely presented, the kernel of α_g is finitely generated by Lemma 1.1.7 (2). As before, pick an element $g' \in R \setminus \mathfrak{m}$ that annihilates all of the generators of $\ker(\alpha_g)$ simultaneously. It follows that $\alpha_{gg'}$ is an isomorphism. Doing this for each maximal ideal $\mathfrak{m}$ gives a collection of elements which generate an ideal that is in no maximal ideal and, hence, must be all of R. This shows (3).

For (4) $\implies$ (5), we only need to show that the assumptions of (4) imply M is finitely generated. First, find an expression $1 = \sum_{i\in I} r_i f_i$ for some nonzero elements $0 \neq r_i \in R$. Any such expression must have only finitely many terms, so we have that I is a finite set. For each index $i \in I$, we can then specify a finite generating set $\{x_{i,j}\}_{j\in J_i}$ for M_{f_i} with the property that each $x_{i,j}$ is in the image of the localization map $M \to M_{f_i}$. If, for each $i \in I$ and $j \in J_i$, we let $\tilde{x}_{i,j} \in M$ denote a preimage of the element $x_{i,j}$, then the collection of all such elements $\{\tilde{x}_{i,j}\}_{i,j} \subset M$ is finite of order say n. Consider the map $\alpha : R^n \to M$ sending the standard basis element e_i to $\tilde{x}_{i,j}$. This map is surjective when localized at any prime ideal $\mathfrak{p} \subset R$ since for each prime ideal $\mathfrak{p} \subset R$ there is at least one element f_i with $f_i \notin \mathfrak{p}$. So, α must already be surjective by [AM69, Proposition 3.8].

Finally, assume (5). We need to show that the assumptions imply M is finitely presented. We use a very similar argument to the one used in the previous step. Since M is finitely generated, there is a short exact sequence

$$0 \to K \to R^{\oplus n} \to M \to 0$$

with K the appropriate kernel. For each prime ideal $\mathfrak{p} \subset R$ the localization of this sequence at $\mathfrak{p}$ is split. Fix one such prime ideal $\mathfrak{p}_i$ and pick an $f_i \in R \setminus \mathfrak{p}_i$ so that M_{f_i} is a free R_{f_i}-module; by Lemma 1.1.7 (1), the module K_{f_i} is finitely generated by elements $e^i_1, ..., e^i_{m_i}$ that we can assume lie in the image of the localization $K \to K_{f_i}$. Doing this at every prime ideal $\mathfrak{p}_i$ gives a collection of elements f_i such that $\sum_i (f_i) = R$. As such, there is a finite subcollection, say $f_1, ..., f_r$ where $(f_1, ..., f_r) = R$. Consider the map

$$\alpha : R^{\oplus \ell} \to K,$$

with $\ell = \sum_{i=1}^r m_i$, defined by sending the standard basis to the elements e^i_j ordered so that $e^i_j < e^{i'}_{j'}$ if either $i < i'$ or $i = i'$ and $j < j'$. The cokernel $\operatorname{coker}(\alpha)$ of this map then vanishes when localized at any prime ideal $\mathfrak{p} \subset R$ so, it must already be trivial by [AM69, Proposition 3.8]. Hence M is also finitely-presented. $\square$

Definition 1.1.12. An R-module M is said to be *locally free of finite rank* if M satisfies any of the equivalent conditions of Lemma 1.1.11. We say that M *has constant rank* r, or we simply say that M *has rank* r, and write $\operatorname{rk}_R(M) = r$ if $\operatorname{rk}_R(M, \mathfrak{p}) = r$ for all prime ideals $\mathfrak{p} \subset R$. When no confusion will occur, we drop the subscript and write $\operatorname{rk}(M)$ for the rank.

Theorem 1.1.13. *Let $R \neq 0$ be an arbitrary ring and let $M \neq 0$ be an R-module. Then M is locally free of finite rank if and only if M satisfies one of the equivalent conditions of Theorem 1.1.10.*

Proof. Suppose that condition (3) of Theorem 1.1.10 holds, i.e. assume that M is finitely presented and flat. Then by Lemma 1.1.8, the module M is both finitely presented and has the property that $M_{\mathfrak{p}}$ is free for all prime ideals $\mathfrak{p} \subset R$, i.e. M satisfies condition (1) of Lemma 1.1.11.

Conversely, if condition (1) of Lemma 1.1.11 holds then it follows that M is finitely presented, by assumption, and M is flat because flatness can be checked locally [AM69, Proposition 3.10], i.e. condition (3) of Theorem 1.1.10 holds. □

Exercises for Section 1.1

1. Let R be a ring and let M be an R-module. Show that the functor $\mathrm{Hom}_R(M, -)$ is left-exact, i.e. for any exact sequence of R-modules

$$0 \to P' \to P \to P'' \to 0,$$

the induced sequence

$$0 \to \mathrm{Hom}_R(M, P') \to \mathrm{Hom}_R(M, P) \to \mathrm{Hom}_R(M, P'')$$

is an exact sequence of R-modules.

Next, prove that an R-module M is projective if and only if the functor $\mathrm{Hom}_R(M, -)$ is also right-exact, i.e. the induced sequence

$$\mathrm{Hom}_R(M, P') \to \mathrm{Hom}_R(M, P) \to \mathrm{Hom}_R(M, P'') \to 0$$

is an exact sequence of R-modules

2. Let R be a ring and let P, P' be two finitely generated and projective R-modules. Prove that $P \otimes_R P'$ is a finitely generated and projective R-module.

3. Let R and S be rings and let $f : R \to S$ be a ring homomorphism giving S the structure of an R-module. Show that if P is a finitely generated and projective R-module, then $P \otimes_R S$ is a finitely generated and projective S-module. Show, moreover, that if $\mathfrak{q} \subset S$ is a prime ideal then $\mathrm{rk}_R(P, f^{-1}(\mathfrak{q})) = \mathrm{rk}_S(P \otimes_R S, \mathfrak{q})$.

4. Let R be a ring and suppose that P and P' are two finitely generated projective R-modules. Let $f : P \to P'$ be a morphism of R-modules.
   (a) Assume that f is surjective and let $K = \ker(f)$. Prove that K is a finitely generated projective R-module and $\mathrm{rk}_R(P, \mathfrak{p}) = \mathrm{rk}_R(K, \mathfrak{p}) + \mathrm{rk}_R(P', \mathfrak{p})$ for any prime ideal $\mathfrak{p} \subset R$.
   (b) Assume that $\mathrm{rk}_R(P, \mathfrak{p}) = \mathrm{rk}_R(P', \mathfrak{p})$ for all prime ideals $\mathfrak{p} \subset R$ and assume that f is a surjection. Show that f is then an isomorphism.

(c) Find an example of a ring R, projective R-modules P, P', and an injection $f : P \to P'$ with the property that P'/P is not projective.

5. Let R be a ring and $I \subset R$ an ideal. Let $S = R/I$. Suppose M is a projective R-module, and let N be any R-module. Show that the canonical map

$$\mathrm{Hom}_R(M, N) \to \mathrm{Hom}_S(M \otimes_R S, N \otimes_R S)$$

is a surjection.

6. (*Open gluing*). Let R be a ring and and let $\{f_i\}_{i \in I} \subset R$ be a collection of elements which generates the unit ideal. Let $m \geq 1$ be an integer and assume that for all pairs $i, j \in I$ there is an element $\phi_{ij} \in \mathrm{GL}_m(R_{f_i f_j})$, i.e. an invertible $m \times m$-matrix with coefficients in $R_{f_i f_j}$, so that the following conditions hold for the collection $\phi = \{\phi_{ij}\}_{i,j \in I}$:
   - for all $i \in I$ we have $\phi_{ii} = I_m$, where I_m is the $m \times m$-identity,
   - $\phi_{jk}\phi_{ij} = \phi_{ik}$ inside $\mathrm{GL}_m(R_{f_i f_j f_k})$ for all triples $i, j, k \in I$.

   Now fix any finite subset $J \subset I$ so that the elements $\{f_i\}_{i \in J}$ also generate the unit ideal and consider the R-submodule

$$P_{\phi,J} \subset \bigoplus_{i \in J} (R_{f_i})^{\oplus m}$$

   defined as the subset

$$P_{\phi,J} = \{(x_i)_{i \in J} \in \bigoplus_{i \in J} (R_{f_i})^{\oplus m} : \phi_{ij}(x_i) = x_j \text{ for all } i, j \in J\}.$$

   (a) Prove that, for any (ϕ, J) as above, the R-module $P_{\phi,J}$ is finitely generated and projective.
   (b) Suppose that $J' \subset I$ is another finite subset so that the elements $\{f_i\}_{i \in J'}$ generate all of R. Prove that there is an isomorphism $P_{\phi,J} \cong P_{\phi,J'}$.
   (c) Suppose that P is an arbitrary finitely generated and projective R-module. Prove that there are elements $\{f_i\}_{i \in I} \subset R$ generating R so that $P \cong P_{\phi,J}$ for some $\phi = \{\phi_{ij}\}_{i,j \in I}$ and some suitable subset $J \subset I$.

7. (*Projective modules and normalization*). Let R be an integral domain and write $F = R_{(0)}$ for the field of fractions of R. Let $\bar{R}$ be the integral closure of R in F and let $I \subset R$ be the ideal $I = \{x \in R : x\bar{R} \subset R\} = \mathrm{Ann}_R(\bar{R}/R)$. It's then immediate to check that I is also an ideal of the ring $\bar{R}$ since if $y \in \bar{R}$ and $x \in I$ then $xy\bar{R} \subset x\bar{R} \subset R$ so that $xy \in I$. The ideal $I \subset R \subset \bar{R}$ is often called the *conductor ideal of R in $\bar{R}$*.

Suppose now that we're given the data of a triple $(P', \bar{P}, \phi)$ where P' is an R/I-module, $\bar{P}$ is a $\bar{R}$-module, and $\phi : P' \otimes_R \bar{R} \to R/I \otimes_R \bar{P}$ is an isomorphism of $\bar{R}/I \cong R/I \otimes_R \bar{R}$-modules. Let $P \subset P' \times \bar{P}$ be the fiber product of R-modules that makes the following diagram Cartesian.

$$\begin{tikzcd}
P \arrow[rr] \arrow[d] & & \bar{P} \arrow[d] \\
P' \arrow[r] & P' \otimes_R \bar{R} \arrow[r, "\phi"] & R/I \otimes_R \bar{P}
\end{tikzcd}$$

Explicitly P is the R-submodule $P = \{(x, y) \in P' \times \bar{P} \mid \phi(x \otimes 1) = 1 \otimes y\}$ of the R-module $P' \times \bar{P}$.

(a) Suppose that $P' = (R/I)^{\oplus n}$ and $\bar{P} = \bar{R}^{\oplus n}$ are both free modules of rank n, and note that ϕ can be understood as an isomorphism between free $\bar{R}/I$-modules of rank n. So, by picking a basis, we can identify $\phi \in \mathrm{GL}_n(\bar{R}/I)$ with an invertible $n \times n$-matrix having coefficients in $\bar{R}/I$.

Assume that there is an isomorphism, with corresponding invertible matrix $\tilde{\phi} \in \mathrm{GL}_n(\bar{R})$, which makes the following diagram commute

$$\begin{tikzcd}
\bar{R}^{\oplus n} \arrow[r, "\tilde{\phi}"] \arrow[d] & \bar{R}^{\oplus n} \arrow[d] \\
(\bar{R}/I)^{\oplus n} \arrow[r, "\phi"] & (\bar{R}/I)^{\oplus n}
\end{tikzcd}$$

where the vertical arrows are the reduction modulo I. Prove that, under these assumptions, P is a free R-module of rank n.

(b) Suppose again that $P' = (R/I)^{\oplus n}$ and $\bar{P} = \bar{R}^{\oplus n}$ are both free of rank n. Use part (a) and the matrix decomposition

$$\begin{pmatrix} \phi & 0 \\ 0 & \phi^{-1} \end{pmatrix} = \begin{pmatrix} I_n & \phi \\ 0 & I_n \end{pmatrix} \begin{pmatrix} I_n & 0 \\ -\phi^{-1} & I_n \end{pmatrix} \begin{pmatrix} I_n & \phi \\ 0 & I_n \end{pmatrix} \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

where I_n is the $n \times n$-identity matrix, to prove that P is, in this case, a finitely generated and projective R-module.

(c) Suppose now, for the general case, that P' and $\bar{P}$ are an arbitrary finitely generated and projective R/I-module and finitely generated and projective $\bar{R}$-module respectively. Prove that P is a finitely generated and projective R-module and show that the canonical R-module homomorphisms

$$P \to P' \quad \text{and} \quad P \to \bar{P}$$

induce isomorphisms

$$P \otimes_R R/I \cong P' \quad \text{and} \quad P \otimes_R \bar{R} \cong \bar{P}$$

of R/I-modules and $\bar{R}$-modules respectively.

(d) Finally, let P_0 be any fixed finitely generated and projective R-module. By setting $P' = R/I \otimes_R P_0$, setting $\bar{P} = P_0 \otimes_R \bar{R}$, and by letting ϕ be the canonical isomorphism

$$P' \otimes_R \bar{R} = R/I \otimes_R P_0 \otimes_R \bar{R} = R/I \otimes_R \bar{P}$$

show that $P_0 \cong P$. Hence every finitely generated and projective R-module arises from this construction for some triple $(P', \bar{P}, \phi)$.

1.2 K-THEORY

We're now in position to introduce the K-theory of a ring R. The definition here will be equivalent to the definition for the K-theory of the affine scheme $\mathrm{Spec}(R)$ introduced in Chapter 2.

Definition 1.2.1. Let R be a commutative ring. Let $P_{fg}(R)$ be the free abelian group on isomorphism classes of finitely generated projective R-modules, i.e. let

$$P_{fg}(R) := \bigoplus_M \mathbb{Z} \cdot M$$

where the index M varies over the choice of a representative for each isomorphism class of finitely generated projective R-module. Let $P_{ex}(R) \subset P_{fg}(R)$ be the subgroup generated by elements $M - L - N$ for each short exact sequence

$$0 \to L \to M \to N \to 0$$

of finitely generated projective R-modules L, M, and N. We define the *K-theory of the ring R* as the quotient group $K(R) = P_{fg}(R)/P_{ex}(R)$.

Although, in the definition of $K(R)$, we have to make a choice of representative for each isomorphism class of finitely generated and projective R-module, we will never distinguish between a finitely generated and projective R-module M and the chosen representative for the isomorphism class of M in practice. Doing so would result in too much mental baggage for only a modicum of truth.

We should point out that if M and N are finitely generated and projective R-modules, then it follows from the canonical exact sequence

$$0 \to M \to M \oplus N \to N \to 0$$

that $[M \oplus N] = [M] + [N]$ inside $K(R)$. On the other hand, there typically is no module which represents the difference $[M] - [N]$.

Additionally, the tensor product $M \otimes_R N$ of finitely generated and projective R-modules M, N is a finitely generated and projective R-module (Exercise 1.1.2). Hence there is an induced endomorphism

$$M \otimes_R - : P_{fg}(R) \to P_{fg}(R) \qquad N \mapsto M \otimes_R N$$

that takes the subgroup $P_{ex}(R)$ into itself due to condition (3) of Theorem 1.1.10. Similar statements hold for $(- \otimes_R M)$. Combined with the naturality of tensor products this gives $K(R)$ the structure of a ring:

Corollary 1.2.2. *Let R be an arbitrary ring. Then the group $K(R)$ is a ring with multiplication induced by the tensor product of R-modules. This multiplication is associative, commutative, and has a unit given by the class $[R]$.* □

The K-theory $K(R)$ of a ring R is a universal object in the following sense: if there is an abelian group A, an assignment of an element $F(M) \in A$ to any finitely generated and projective R-module M, and if $F(-)$ is additive on short exact sequences, then there is a homomorphism from $K(R)$ to A which is initial among all quotients of $P_{fg}(R)$ admitting morphisms to A sending M to $F(M)$.

Example 1.2.3. Let R be any ring and let $\pi_0(R)$ be the collection of connected components of $\mathrm{Spec}(R)$. For each of the $X_i \in \pi_0(R)$, choose a prime ideal $\mathfrak{p}_i \in X_i$. For any finitely generated and projective R-module M, the localization $M_{\mathfrak{p}_i}$ is free and of rank $\mathrm{rk}(M_{\mathfrak{p}_i}) \in \mathbb{Z}_{\geq 0}$ because of Lemma 1.1.8.

Define a homomorphism

$$\mathrm{rk} : P_{fg}(R) \to \mathbb{Z}^{\oplus \pi_0(R)} \quad \text{by} \quad M \mapsto \sum \mathrm{rk}(M_{\mathfrak{p}_i}) e_i$$

where e_i is the standard basis element associated to the component $X_i \in \pi_0(R)$. Then, for any short exact sequence

$$0 \to L \to M \to N \to 0$$

of finitely generated and projective R-modules L, M, and N one has

$$\mathrm{rk}(M_{\mathfrak{p}_i}) - \mathrm{rk}(L_{\mathfrak{p}_i}) - \mathrm{rk}(N_{\mathfrak{p}_i}) = 0.$$

It follows that the homomorphism rk descends to a (surjective) homomorphism

$$\mathrm{rk} : K(R) \to \mathbb{Z}^{\oplus \pi_0(R)}$$

which we call *the rank homomorphism for $K(R)$*.

Remark 1.2.4. For any ring R, the rank homomorphism from $K(R)$ that was defined in Example 1.2.3 was dependent on a choice of base points $\mathfrak{p}_i \in \operatorname{Spec}(R)$. But, because of Lemma 1.1.11 (5), any choice of base points defines the same homomorphism; in this way we're justified in calling it *the* rank homomorphism.

Example 1.2.5. Suppose that R is a PID. By the fundamental theorem of finitely generated modules over a PID, any finitely generated R-module M is isomorphic to a direct sum

$$M = R^{\oplus n} \oplus R/(d_1) \oplus \cdots \oplus R/(d_m)$$

for some elements $d_1, ..., d_m \in R$. If M is moreover projective, then each summand of M must also be finitely generated and projective. In particular, this implies that any finitely generated and projective R-module is free and, from this fact, we can compute $K(R)$.

However, we could already compute $K(R)$ even without observing that the R-module $R/(d_i)$ is not projective (when d_i is not a unit). If the R-modules $R/(d_i)$ were projective, then from the exact sequences

$$0 \to R \xrightarrow{\cdot d_i} R \to R/(d_i) \to 0$$

for each $1 \leq i \leq m$, it follows that $[R/(d_i)] = [R] - [R] = 0$. Hence the equality

$$[M] = [R^{\oplus n}] + [R/(d_1)] + \cdots + [R/(d_m)] = n[R]$$

inside of $K(R)$. Since $\operatorname{rk}([R]) \neq 0$, it follows that $K(R) = \mathbb{Z}$.

The universal nature of $K(R)$ is what makes it both an interesting and difficult object to study for most rings R. In the remainder of this section, we use this universality to relate the K-theory $K(R)$ with the Picard group $\operatorname{Pic}(R)$, another object from algebraic geometry which is both interesting and difficult to study. Along the way, we work out some examples which illustrate how knowledge of the structure of the Picard group can be used to see that $K(R)$ is typically nontrivial.

Definition 1.2.6. Let R be any commutative ring. An R-module M is said to be an *invertible R-module* if M is a locally free module of finite rank and for every prime ideal $\mathfrak{p} \subset R$ one has $\operatorname{rk}(M_\mathfrak{p}) = 1$.

Lemma 1.2.7. *Let R be a commutative ring. Then the following statements hold.*

(1) *For two invertible R-modules M and N, the tensor product $M \otimes_R N$ is an invertible R-module.*
(2) *If M is an invertible R-module, then $\operatorname{Hom}_R(M, R)$ is an invertible R-module.*

Proof. An invertible R-module is both finitely generated and projective because of Theorem 1.1.13, the tensor product of two finitely generated projective modules is again finitely generated and projective and, for any prime $\mathfrak{p} \subset R$, the localization

$$(M \otimes_R N)_{\mathfrak{p}} = M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} N_{\mathfrak{p}}$$

is a free $R_{\mathfrak{p}}$-module of rank $\mathrm{rk}(M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} N_{\mathfrak{p}}) = \mathrm{rk}(R_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} R_{\mathfrak{p}}) = 1$. This proves (1).

For the second claim, we have by Theorem 1.1.13 that for any given invertible R-module M there exists an R-module N with $M \oplus N \cong R^{\oplus n}$ for some $n > 0$. From the isomorphisms

$$R^{\oplus n} \cong \mathrm{Hom}_R(R^{\oplus n}, R) \cong \mathrm{Hom}_R(M \oplus N, R) \cong \mathrm{Hom}_R(M, R) \oplus \mathrm{Hom}_R(N, R)$$

we find that $\mathrm{Hom}_R(M, R)$ also satisfies the equivalent conditions of Theorem 1.1.10. In particular, $\mathrm{Hom}_R(M, R)$ is locally free of finite rank. Now, using Lemma 1.1.9, we see that for any prime $\mathfrak{p} \subset R$ the $R_{\mathfrak{p}}$-module

$$\mathrm{Hom}_R(M, R)_{\mathfrak{p}} = \mathrm{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, R_{\mathfrak{p}})$$

has rank $\mathrm{rk}(\mathrm{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, R_{\mathfrak{p}})) = \mathrm{rk}(\mathrm{Hom}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}, R_{\mathfrak{p}})) = 1$. This proves (2). □

The justification for calling an R-module M an invertible module is explained by the following lemma.

Lemma 1.2.8. *Let R be any commutative ring. Let M be an invertible R-module. Then the homomorphism*

$$\mathrm{Hom}_R(M, R) \otimes_R M \to R \quad \textit{defined by} \quad f \otimes m \mapsto f(m)$$

is an isomorphism.

Proof. For any prime ideal $\mathfrak{p} \subset R$, the given homomorphism localizes over $\mathfrak{p}$ to give a morphism

$$\mathrm{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, R_{\mathfrak{p}}) \otimes_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \to R_{\mathfrak{p}}$$

between free $R_{\mathfrak{p}}$-modules of the same rank. The localized morphism is surjective, and therefore an isomorphism; the result follows from [AM69, Proposition 3.9]. □

Definition 1.2.9. Let R be any commutative ring. Write $\mathrm{Pic}(R)$ for the set of isomorphism classes of invertible R-modules. The R-module tensor product gives $\mathrm{Pic}(R)$ the structure of an abelian group which we call *the Picard group of R*.

The K-theory of R is naturally related to the Picard group of R. To see this, we introduce the determinant of a locally free R-module.

Definition 1.2.10. Let R be a commutative ring and M an R-module. The nth exterior product $\wedge^n M$, for any integer $n \geq 1$, is the quotient of nth tensor power $M^{\otimes n}$ by the R-submodule generated by elements $m_1 \otimes \cdots \otimes m_n$ with $m_i = m_j$ for some $i \neq j$. If $m_1 \otimes \cdots \otimes m_n$ is any simple tensor, then we write $m_1 \wedge \cdots \wedge m_n$ for its image in $\wedge^n M$. If M is a locally free R-module of constant finite rank $\mathrm{rk}(M) = r$, then the *determinant of* M is the rth exterior product $\det(M) = \wedge^r M$.

If M is a free R-module of finite rank $\mathrm{rk}(M) = r$, then for any $1 \leq k \leq r$ the exterior product $\wedge^k M$ is also free of rank $\mathrm{rk}(\wedge^k M) = \binom{r}{k}$. For all $k > r$, we have $\wedge^k M = 0$ and, by convention, we set $\wedge^k M = 0$ if $k < 0$ and $\wedge^0 M = R$.

Since the formation of exterior products commutes with localization (i.e. since $(\wedge^k M)_f = \wedge^k(M_f)$ for all f in R), the same statements hold for any R-module M that is locally free of finite rank. In particular, if M is locally free of constant finite rank $\mathrm{rk}(M) = r$, then $\det(M)$ is locally free of finite rank $\mathrm{rk}(\det(M)) = 1$. Hence $\det(M)$ is an invertible R-module.

Lemma 1.2.11. *Let R be any commutative ring and let*

$$0 \to L \to M \to N \to 0$$

be a short exact sequence of finite rank locally free R-modules. Then there is a filtration $F_\bullet(\wedge^n M)$ of $\wedge^n M$ by R-submodules

$$\wedge^n L = F_n(\wedge^n M) \subset F_{n-1}(\wedge^n M) \subset \cdots \subset F_0(\wedge^n M) = \wedge^n M$$

and isomorphisms

$$\psi_i : \wedge^i L \otimes \wedge^{n-i} N \xrightarrow{\sim} F_i(\wedge^n M)/F_{i+1}(\wedge^n M)$$

for all $0 \leq i \leq n$.

Proof. For each $i \geq 0$, define $F_i(\wedge^n M)$ to be the R-submodule of $\wedge^n M$ generated by the elements $y_1 \wedge \cdots \wedge y_i \wedge z_1 \wedge \cdots \wedge z_{n-i}$ where the y_j belong to $L \subset M$ and z_j are arbitrary. Let $\tilde{x}_1, ..., \tilde{x}_k$ generate N and choose respective preimages $x_1, ..., x_k$ in M. Define a map

$$\psi_i : \wedge^i L \otimes \wedge^{n-i} N \to F_i(\wedge^n M)/F_{i+1}(\wedge^n M)$$

by the formula

$$\psi_i(a_1 \wedge \cdots \wedge a_i \otimes \tilde{b}_1 \wedge \cdots \wedge \tilde{b}_{n-i}) = a_1 \wedge \cdots \wedge a_i \wedge b_1 \wedge \cdots \wedge b_{n-i}$$

where if $\tilde{b}_k = \sum c_j \tilde{x}_j$ then $b_k = \sum c_j x_j$. The map ψ_i is well-defined since if $\tilde{b}_k = \sum d_j \tilde{x}_j$ is another expression for $\tilde{b}_k$ then from the relation (in N)

$$0 = \sum (c_j - d_j)\tilde{x}_j$$

we find that $0 = \sum (c_j - d_j)x_j$ inside M by localizing at all prime ideals $\mathfrak{p} \subset R$. We won't need it but, the map ψ_i is canonical in that it also doesn't depend on the choice of lifts $x_1, ..., x_k$ since any other element x'_j lifting $\tilde{x}_j$ has the property that $x'_j - x_j$ is contained in L.

Now the fact that ψ_i is an isomorphism can be seen locally. After localization at any prime ideal $\mathfrak{p} \subset R$, the morphism ψ_i is a surjection between free $R_\mathfrak{p}$-modules of the same finite rank. □

As immediate corollaries we get:

Corollary 1.2.12. *Let R be a commutative ring, and let*

$$0 \to L \to M \to N \to 0$$

be a short exact sequence of finite rank locally free sheaves of constant rank, so $n = \mathrm{rk}(M)$, $k = \mathrm{rk}(L)$ and $n - k = \mathrm{rk}(N)$. Then $\det(M) = \det(L) \otimes \det(N)$.

Proof. For all $i \neq k$, the module $\wedge^i L \otimes \wedge^{n-i} N = 0$ by rank considerations. In the notation of Lemma 1.2.11, this implies $F_i(\wedge^n M) = \wedge^n M$ if $i \leq k$, $F_i(\wedge^n M) = 0$ if $i > k$, and ψ_k gives an isomorphism between $\det(M)$ and $\det(L) \otimes \det(N)$. □

Corollary 1.2.13. *Let R be any commutative ring and assume that R has a unique minimal prime ideal. Then the determinant extends to a group homomorphism*

$$\det : K(R) \to \mathrm{Pic}(R)$$

defined by $\det([M]) = [\det(M)]$.

Proof. If $\mathfrak{p} \subset R$ is the unique minimal prime ideal of R, then for any finite rank locally free R-module M, the rank function $\mathrm{rk}(M, -) : \mathrm{Spec}(R) \to \mathbb{Z}$ of Lemma 1.1.11 is constant and determined by the value $\mathrm{rk}(M, \mathfrak{p})$. In particular, this means that for any short exact sequence of locally free R-modules of finite rank

$$0 \to L \to M \to N \to 0,$$

the conditions of Corollary 1.2.12 are satisfied (compare with Example 1.2.3). Hence the assignment sending a locally free R-module M to $\det(M)$ descends to a morphism from $K(R)$. □

The determinant homomorphism of Corollary 1.2.13 is always a surjection since the determinant of an invertible R-module M is M itself. By the same reasoning, there is an injective homomorphism from $\mathrm{Pic}(R)$ to the group of units of the ring $K(R)$, sending the class of an invertible module to itself, giving $\mathrm{Pic}(R) \subset K(R)^\times$. In these two ways, we can view $K(R)$ as a more complicated object than $\mathrm{Pic}(R)$.

Nonetheless, it's already a difficult problem to find examples of rings R where $\mathrm{Pic}(R)$ can be computed precisely. The following is an example where $\mathrm{Pic}(R) \neq 0$; the proof here is from Lam's book on rings and modules [Lam99, Example 2.19B].

Example 1.2.14. Let k be any field such that -1 isn't the square of any element from k (e.g. we could have $k = \mathbb{R}$; note also that the characteristic of k isn't 2). Set $R = k[x,y]/(x^2 + y^2 - 1)$ and let $(x, y-1) = I \subset R$ be the ideal generated by the two elements x and $y - 1$. We'll show here that I is a nontrivial invertible R-module, and hence $\mathrm{Pic}(R) \neq 0$. (In fact we'll see, in Example 3.2.1 below, that $\mathrm{Pic}(R) \cong \mathbb{Z}/2\mathbb{Z}$ with $[I]$ being the nontrivial generator). It follows from Corollary 1.2.13 that the class $[I] \in K(R)$ is nonzero (and also $[I] \neq n[R]$ for any $n \in \mathbb{Z}$).

To start, we observe that R is a domain. Indeed, the element $y-1$ is irreducible inside $k[y]$, and hence prime since $k[y]$ is a unique factorization domain (UFD). So, if we considered the element $q(x) = x^2 + y^2 - 1$ inside $k[y][x] = k[x, y]$ then we could apply Eisenstein's criterion [DF04, §9.4, Proposition 13] (noting that $y^2 - 1$ is contained in the ideal $(y-1)$ but not the ideal $(y-1)^2$ of $k[y]$) to see that $q(x)$ is irreducible inside $k[x, y]$. Since $k[x, y]$ is also a UFD, it follows that R is a domain as claimed.

We can show that $I = (x, y - 1)$ is an invertible R-module by utilizing Lemma 1.1.11. That is to say, we have $R = (y - 1, y + 1)$ and if we can find isomorphisms

$$I_{y-1} \cong R_{y-1} \quad \text{and} \quad I_{y+1} \cong R_{y+1}$$

then the lemma says that I is an invertible R-module. For the first isomorphism, we observe that the inclusion $I \subset R$ becomes an isomorphism after localization at the multiplicative subset generated by $y - 1$. For the second isomorphism, it suffices to note that I_{y+1} is a principal ideal of R_{y+1} generated by $x/(y + 1)$.

To see that I is a nontrivial invertible R-module, we assume for a contradiction that there is an isomorphism $R \cong I$ and denote by $f \in I$ a principal generator for this R-module. This means that we can find elements $g, h \in R$ and equalities

$$fg = x \quad \text{and} \quad fh = y - 1. \tag{1.2.15}$$

If we denote the fraction field of R by $F = R_{(0)}$, then the inclusion $k[y] \subset R$ induces an inclusion $k(y) \subset F$ realizing F as a finite algebraic extension of $k(y)$. More precisely, the ring map

$$\phi : k(y)[x] \to F \quad \text{defined by} \quad x \mapsto x$$

has kernel containing the ideal generated by x^2+y^2-1. By Gauss's lemma [DF04, §9.3, Proposition 5], using that $k[y]$ is a UFD, the polynomial $x^2 + y^2 - 1$ remains irreducible in $k(y)[x]$ so that the quotient $k(y)[x]/(x^2+y^2-1)$ is a finite extension of the field$k(y)$. Since R is contained in the image $\mathrm{Im}(\phi)$, which is a field, it follows that $F \cong k(y)[x]/(x^2 + y^2 - 1)$. Now we can use the existence of the field norm

$$N_{F/k(y)} : F^\times \to k(y)^\times$$

along with the equalities of (1.2.15) to get a contradiction. (If you haven't seen it before, the field norm is defined by sending an element $a \in F^\times$ to the determinant $\det(m_a)$ of the $k(y)$-linear transformation $m_a : F \to F$ which sends an element x to the product $m_a(x) = ax$. The field norm is a group homomorphism since

$$\det(m_1) = 1 \quad \text{and} \quad \det(m_{a/b}) = \det(m_a \circ m_{1/b}) = \det(m_a) \cdot \det(m_b)^{-1}$$

for all $a, b \in F^\times$.)

The ring R is a free $k[y]$-module with generators 1 and x so we can write $f = f_0 + f_1 x$ for some elements $f_0, f_1 \in k[y]$. Taking the norm of f shows

$$N_{F/k(y)}(f) = N_{F/k(y)}(f_0 + f_1 x) = f_0^2 - x^2 f_1^2 = f_0^2 - (1 - y^2) f_1^2 = f_0^2 + y^2 f_1^2 - f_1^2$$

with the latter a nonzero polynomial in y of even degree (this is where we use the assumption that -1 isn't the square of an element from k, so the leading terms of f_0^2 and $y^2 f_1^2$ don't cancel). However, taking the norm of the expressions from (1.2.15) and subtracting gives

$$\begin{aligned} N_{F/k(y)}(f)\left(N_{F/k(y)}(g) - N_{F/k(y)}(h)\right) &= N_{F/k(y)}(fg) - N_{F/k(y)}(fh) \\ &= N_{F/k(y)}(x) - N_{F/k(y)}(y-1) \\ &= -x^2 - (y-1)^2 \\ &= -(1-y^2) - (y-1)^2 \\ &= 2(y-1). \end{aligned}$$

Since $2(y-1)$ can't be a multiple (inside $k[y]$) of the nonzero polynomial $N_{F/k(y)}(f)$ of even degree in y, we've reached a contradiction to our assumption $R \cong I$.

Example 1.2.16. In this example, we show how the computation of Example 1.2.14 can be used to get examples of higher dimension as well. Let k be any field with $-1 \notin k^{\times 2}$ as before. Set $R = k[x, y, z]/(xy - z^2 + 1)$ and let $(x, z-1) = I \subset R$ be the ideal generated by the two elements x and $z - 1$. As in Example 1.2.14, we can show that I is a nontrivial invertible R-module, so that $\operatorname{Pic}(R) \neq 0$. Later on, in Exercise 1.3.7, we'll show that $\operatorname{Pic}(R) \cong \mathbb{Z}$ with I a nontrivial generator. In this case $X = \operatorname{Spec}(R)$ is a one-sheeted hyperboloid in $\mathbb{A}^3$ and the closed subscheme $L = \operatorname{Spec}(R/I)$ of X is the line in the y-direction at $x = 0$ and $z = 1$.

Before we begin proving the above statements, we first note that R is a domain. Indeed, the element $xy + 1$ is an irreducible element $k[x, y]$, which can be checked directly, so the ideal $(xy + 1)$ is prime in $k[x, y]$ since this latter ring is a unique factorization domain (UFD). The polynomial $xy - z^2 + 1$, considered as an element in $k[x, y][z]$, is therefore irreducible by Eisenstein's criterion. As $k[x, y, z]$ is also a UFD, this proves that R is a domain as claimed.

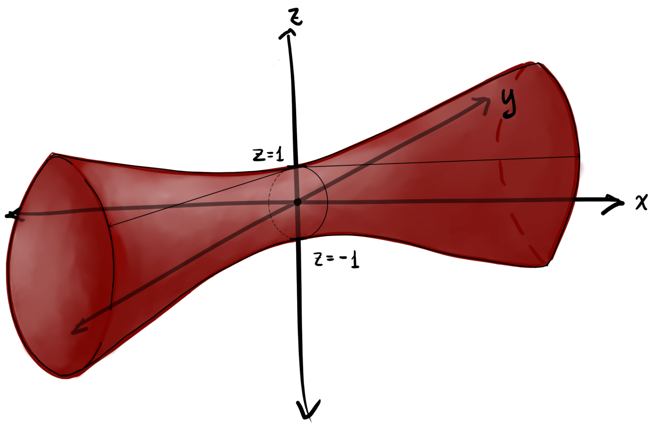

Figure 1.1: The vanishing locus $V(xy - z^2 + 1)$ inside $\mathbb{A}^3$

To see that I is an invertible R-module, we utilize Lemma 1.1.11. That is, we observe $R = (z-1, z+1)$ so that it suffices to find isomorphisms

$$I_{z-1} \cong R_{z-1} \quad \text{and} \quad I_{z+1} \cong R_{z+1}.$$

The inclusion $I \subset R$ becomes an isomorphism after localizing at the multiplicative set generated by $z-1$, handling this case. For the other isomorphism, we note that inside the domain R_{z+1} the ideal I_{z+1} is principal and generated by $x/(z+1)$.

Lastly, we need to show that I isn't itself isomorphic with R. Let $J \subset R$ be the ideal $J = (x+y)$ and write $S = R/J$. Then we have isomorphisms

$$S = k[x,y,z]/(xy - z^2 + 1, x+y) \cong k[x,z]/(x^2 + z^2 - 1).$$

The S-module $I \otimes_R S$ is invertible and isomorphic with the ideal $\bar{I} = (x, z-1) \subset S$. Indeed, the composition

$$I \otimes_R S \to R \otimes_R S \xrightarrow{a \otimes b \mapsto ab} S$$

has image $\bar{I}$, so that there is a surjection $\psi : I \otimes_R S \to \bar{I}$. Localized at any prime ideal $\mathfrak{p} \subset S$, the map $\psi_{\mathfrak{p}}$ is a surjection between free $S_{\mathfrak{p}}$-modules of rank one. This implies that ψ is an isomorphism locally, hence also globally. According to Example 1.2.14, the ideal $\bar{I}$ isn't isomorphic with S, completing the proof.

Exercises for Section 1.2

1. Assume that $(R, \mathfrak{m})$ is a nonzero local ring. Find an isomorphism $K(R) \cong \mathbb{Z}$.

2. Let k be a field and t an indeterminate. Determine each of the rings $K(k[t])$, $K(k[t, t^{-1}])$, and $K(k[[t]])$.

3. Let $f : R \to S$ be a homomorphism between rings R and S. Use Exercise 1.1.3 to show that the assignment

$$\mathrm{res}_R^S : K(R) \to K(S) \qquad [M] \mapsto [M \otimes_R S]$$

is a well-defined ring homomorphism. Show also that if $g : S \to T$ is another ring homomorphism, then $\mathrm{res}_S^T \circ \mathrm{res}_R^S = \mathrm{res}_R^T$.

4. Let R be a commutative ring. This exercise shows that the group of relations $P_{ex}(R) \subset P_{fg}(R)$ can be modified without changing the quotient $K(R)$.
   (a) Show that a long exact sequence of finitely generated projective R-modules

$$0 \to N_m \to \cdots \to N_1 \to 0$$

   can be split into a collection of short exact sequences

$$0 \to K_i \to N_i \to K_{i-1} \to 0$$

   with both K_i and K_{i-1} finitely generated and projective R-modules (use Exercise 1.1.4).
   (b) Let $P_{lex}(R) \subset P_{fg}(R)$ be the subgroup generated by elements

$$\sum_{i \geq 1} (-1)^i N_i$$

   for every long exact sequence as above. Define $K'(R) := P_{fg}(R)/P_{lex}(R)$. Show that the canonical morphism

$$K(R) \to K'(R) \qquad [M] \mapsto [M]$$

   is an isomorphism.

5. Let R be a commutative ring and define $P_{inf}(R)$ as the free abelian group generated by isomorphism classes of all projective R-modules (not necessarily finitely generated). Define $P_{ex,inf}(R)$ to be the subgroup of $P_{inf}(R)$ generated by those elements $M - N - L$ coming from short exact sequences

$$0 \to L \to M \to N \to 0$$

of projective R-modules. Show that $[R] = 0$ in the quotient $P_{inf}(R)/P_{ex,inf}(R)$.

6. Let R be a ring with $\mathrm{Spec}(R)$ having a connected underlying topological space. Prove that if P is a finite rank locally free R-module, then P has constant rank. Verify that the morphism $\det : K(R) \to \mathrm{Pic}(R)$ defined as in Corollary 1.2.13 is well-defined for such rings. Extend the construction of the determinant map to an arbitrary ring R with possibly disconnected spectrum $\mathrm{Spec}(R)$.

7. Suppose $f : R \to S$ is a homomorphism of rings. Show that the assignment

$$\mathrm{res}_R^S : \mathrm{Pic}(R) \to \mathrm{Pic}(S) \qquad [I] \mapsto [I \otimes_R S]$$

is a well-defined group homomorphism with the property that, if $g : S \to T$ is another ring homomorphism then $\mathrm{res}_S^T \circ \mathrm{res}_R^S = \mathrm{res}_R^T$. Show that, moreover, these morphisms fit into a commutative square

$$\begin{tikzcd} K(R) \arrow[r, "\mathrm{res}_R^S"] \arrow[d, "\det"] & K(S) \arrow[d, "\det"] \\ \mathrm{Pic}(R) \arrow[r, "\mathrm{res}_R^S"] & \mathrm{Pic}(S) \end{tikzcd}$$

with the morphisms from Exercise 1.2.3.

8. Let R be any commutative ring. For any integer $n \geq 0$, we write $M_n(R)$ for the ring of $n \times n$-matrices with coefficients in R. An element e in a (possibly noncommutative) ring S is an idempotent if there is an equality $e^2 = e$.
   (a) Show that every idempotent $e \in M_n(R)$ determines a finitely generated projective R-module P_e as the image submodule $P_e = e(R^{\oplus n}) \subset R^{\oplus n}$. Conversely, every finitely generated projective R-module P is isomorphic to some P_e for some idempotent e in $M_n(R)$ for some $n \geq 1$.
   (b) Assume that $I \subset R$ is an ideal contained in the Jacobson radical of R, i.e. I is an ideal contained in every maximal ideal of R. Use Exercise 1.1.5 to show that if M and N are two finitely generated projective R-modules which admit a surjection (resp. an isomorphism) $\bar{f} : M/IM \to N/IN$ of R/I-modules, then there is a surjection (resp. an isomorphism) $f : M \to N$ of R-modules which reduces modulo I to $\bar{f}$.
   (c) Suppose that I is also nilpotent, so $I^N = 0$ for some large $N \geq 1$, and let $e \in M_n(R/I)$ be a given idempotent determining a finitely generated and projective R/I-module P_e. Prove there exists an idempotent $\tilde{e} \in M_n(R)$ so that $\tilde{e} \equiv e \pmod{I}$. The R-module $P_{\tilde{e}}$ determined by any such lift $\tilde{e}$ has the property that $P_{\tilde{e}} \otimes_R R/I \cong P_e$.
   (Hint: let f be any lift of e to $M_n(R)$. Then $(f^2 - f)^N = 0$ so that

$$0 = \sum_{k=0}^{N} \binom{N}{k} (f^2)^k (-f)^{N-k} = \sum_{k=0}^{N} (-1)^{N-k} \binom{N}{k} f^{N+k}.$$

Setting $-h = \sum_{k=1}^{N}(-1)^{N-k}\binom{N}{k} f^{k-1}$ gives $0 = f^N - f^{N+1}h$. Now if we set $\tilde{e} = f^N h^N$ then since $hf = fh$ we have

$$\tilde{e}^2 = f^{2N}h^{2N} = f^{N-1}(f^{N+1}h)h^{2N-1} = f^{2N-1}h^{2N-1} = \cdots = f^N h^N = \tilde{e}.$$

It remains to show that $\tilde{e} = e$ modulo I.)

(d) Use parts (a) - (c) above to show that, if R is any commutative ring and if $I \subset R$ is any nilpotent ideal, then the map

$$\mathrm{res}_R^{R/I} : K(R) \to K(R/I)$$

from Exercise 1.2.3 is an isomorphism.

(e)* Extend part (c) above and show that, if R is any commutative ring which is complete with respect to an ideal I, then the map

$$\mathrm{res}_R^{R/I} : K(R) \to K(R/I)$$

from Exercise 1.2.3 is an isomorphism.

9. Suppose that we're in the set-up of Exercise 1.1.7, i.e. we have $(R, \bar{R}, F, I)$ with R a domain, $\bar{R}$ the integral closure of R in the fraction field $F = R_{(0)}$ and I the conductor ideal of $R \subset \bar{R}$. Prove that there is an exact sequence

$$1 \to R^\times \to \bar{R}^\times \times (R/I)^\times \to (\bar{R}/I)^\times \to \mathrm{Pic}(R) \to \mathrm{Pic}(\bar{R}) \times \mathrm{Pic}(R/I) \to \mathrm{Pic}(\bar{R}/I)$$

where the maps are defined as:
- $R^\times \to \bar{R}^\times \times (R/I)^\times$ sends x to (x, x)
- $\bar{R}^\times \times (R/I)^\times \to (\bar{R}/I)^\times$ sends (x, y) to xy^{-1}
- $(\bar{R}/I)^\times \to \mathrm{Pic}(R)$ sends a unit x to the class of the invertible R-module constructed as in Exercise 1.1.7 for the triple $(R/I, \bar{R}, x)$
- $\mathrm{Pic}(R) \to \mathrm{Pic}(\bar{R}) \times \mathrm{Pic}(R/I)$ sends $[I]$ to the pair $\left(\mathrm{res}_R^{\bar{R}}([I]), \mathrm{res}_R^{R/I}([I])\right)$
- $\mathrm{Pic}(\bar{R}) \times \mathrm{Pic}(R/I) \to \mathrm{Pic}(\bar{R}/I)$ sends $([I], [J])$ to $\mathrm{res}_{\bar{R}}^{\bar{R}/I}([I]) - \mathrm{res}_{R/I}^{\bar{R}/I}([J])$.

10. Let k be a field and let R be the coordinate ring $R = k[x, y]/(y^2 - x^3)$ of the affine cuspidal cubic curve.
   (a) Show that the k-algebra homomorphism $R \to k[t]$ defined by sending x to t^2 and y to t^3 is an injection with image the subring $k[t^2, t^3] \subset k[t]$. Argue that $k[t]$ is the integral closure of R inside the fraction field $k(t)$.
   (b) Use Exercise 1.2.9 to construct an isomorphism of groups $\mathrm{Pic}(R) \cong k$. Given an element $a \in k$, can you describe the isomorphism class of the invertible R-module corresponding to a?

11. Let k be a field and let R be the coordinate ring $R = k[x, y]/(y^2 - x^3 - x^2)$ of the affine nodal cubic curve.
    (a) Show that the k-algebra homomorphism $R \to k[t]$ which sends x to $t^2 - 1$ and y to $t^3 - t$ is an injection with image the subring $k[t^2 - 1, t^3 - t] \subset k[t]$. Argue that $k[t]$ is the integral closure of R inside the fraction field $k(t)$.
    (b) Use Exercise 1.2.9 to construct an isomorphism of groups $\mathrm{Pic}(R) \cong k^\times$. Given an element $u \in k^\times$, can you describe the isomorphism class of the invertible R-module corresponding to u?

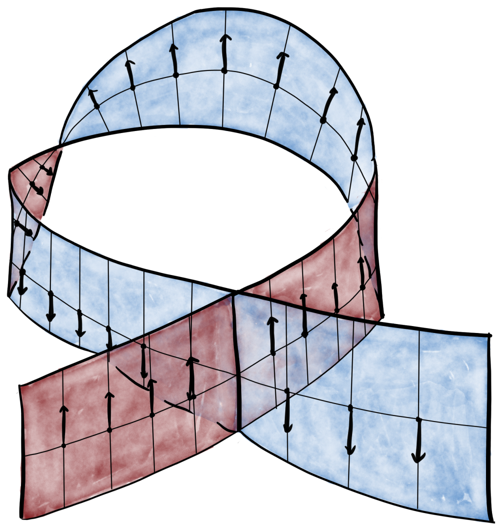

Figure 1.2: The line bundle corresponding to $-1 \in k^\times$ on the nodal cubic

1.3 Divisors

In this section, we develop the algebraic theory of divisors for some types of rings. (Depending on the type of divisor, the assumptions that we add to the ring will be more-or-less restrictive. In the broadest setting, we work with integral domains; in the most restrictive setting, we focus on integrally closed Noetherian domains.) Essentially all of the terminology and ideas presented here will be carried over to the setting of a more general scheme later in this book and, when we do this, it will be clear that the definitions and constructions of this section are simply the algebraic formulations for the theory in the case of the corresponding affine scheme. However, most of the subtle nuances of the theory can already be understood through study of this commutative algebra.

There are two types of divisors that we focus on in this book. The first type that appear below are called Cartier divisors which, in the case of rings, are equivalent with certain modules called fractional ideals. Instead of giving the abstract (and unmotivated) definition of a Cartier divisor, and building the theory from there, we start from the equivalent notion of a fractional ideal and we derive the relationship between these modules and the Picard group in the case of an integral domain. The formal definition of a Cartier divisor, which forms the basis for the theory in the case of a general scheme, is given in the exercises along with some exploration of the equivalence to fractional ideals.

The second type of divisor appearing in this section is called a Weil divisor. These divisors are intimately connected with geometry and, even in the case of an affine scheme, it's difficult to formulate the theory without relying on the geometric ideas from which they originate. In order to simplify some considerable amount of technical detail, we will typically work with Weil divisors only in a more restricted setting (e.g. Noetherian domains). Here we introduce and study the divisor class group, which is a kind of analog of the Picard group defined using Weil divisors (although, the divisor class group and the Picard group convey a considerable amount of inherently different information). We end by comparing the two notions of Weil divisors and Cartier divisors (in our guise of fractional ideals), by comparing the Picard group to the divisor class group, and by giving some applications.

Fractional ideals and the Picard group

Definition 1.3.1. Let R be an integral domain and let $F = R_{(0)}$ be the field of fractions of R. An R-submodule $I \subset F$ is called a *fractional ideal for* R if $I \neq 0$ and if there exists an element $f \in R \setminus \{0\}$ giving containment

$$fI = \{fg \in F : g \in I\} \subset R.$$

Any nonzero ideal $J \subset R$ can be considered a fractional ideal via the inclusion $J \subset R \subset F$; a fractional ideal I is called an *integral fractional ideal* if it is the image of an ideal $J \subset R$.

Remark 1.3.2. If R is a Noetherian domain with fraction field F, then an R-submodule $I \neq 0$ of F is a fractional ideal for R if and only if I is finitely generated.

If I and J are two fractional ideals of an integral domain R, then one can define *the fractional ideal product of* I *and* J as

$$IJ = \left\{\sum f_i g_i : f_i \in I,\ g_i \in J\right\} \subset F.$$

This is also a fractional ideal for R (if $f_1 I \subset R$ and $f_2 J \subset R$ then $f_1 f_2 IJ \subset R$). It's easy to check that the fractional ideal product is associative and commutative. Moreover if $I, J \subset R$ are two ideals, then the fractional ideal product of I and J agrees with the ideal product of I and J.

For a fractional ideal $I \subset F$ for R, one can also define a fractional ideal inverse

$$I^{-1} = \{f \in F : fI \subset R\}$$

as the largest R-submodule of F such that $I^{-1}I \subset R$.

Lemma 1.3.3. *Let R be a integral domain with $F = R_{(0)}$ its field of fractions. Then the following statements are true.*

(1) *If $I \subset F$ is a fractional ideal for R, then I^{-1} is a fractional ideal for R.*
(2) *If $I \subset F$ is a fractional ideal for R, then $IR = I = RI$.*

Proof. To see that I^{-1} is a fractional ideal for R, let $f \in R \setminus \{0\}$ be an element such that $fI \subset R$. Since fI is an R-submodule of R, we have that $fI = (a_1, a_2, ...)$ is an ideal of R generated by some elements $a_1, a_2, ... \in R$. We claim that $a_1 I^{-1} \subset R$. Indeed, for any $g \in I^{-1}$ we have $gI \subset R$ so that $(a_1 f^{-1})g \in R$ as $a_1 f^{-1} \in I$. Hence

$$a_1 g = (a_1 f^{-1}) f g \in R.$$

Since g was arbitrary, we find $a_1 I^{-1} \subset R$ as claimed.

For (2) we note $I = 1 \cdot I \subset RI \subset I$ as I is an R-module. Similarly $RI = I$. $\square$

A fractional ideal $I \subset F$ for R is called an *invertible fractional ideal* if there is an equality $II^{-1} = R = I^{-1}I$. If I is an invertible fractional ideal, then so is I^{-1}. Indeed, since $II^{-1} \subset R$ we have $I \subset (I^{-1})^{-1}$. But we also have

$$(I^{-1})^{-1} = R(I^{-1})^{-1} = (II^{-1})(I^{-1})^{-1} = I(I^{-1}(I^{-1})^{-1}) \subset IR = I$$

so that $I = (I^{-1})^{-1}$ and $I^{-1}(I^{-1})^{-1} = R$. A similar argument shows if $I, J \subset F$ are two invertible fractional ideals, then $(IJ)^{-1} = I^{-1}J^{-1}$ and the fractional ideal product IJ is invertible as well.

If we denote by $I_{fr}(R)$ the set of all invertible fractional ideals for the integral domain R, then the above shows that $I_{fr}(R)$ has the structure of an abelian group with the product of two fractional ideals given by the fractional ideal product. We'd like to now compare the group $I_{fr}(R)$ of all invertible fractional ideals with the group $\mathrm{Pic}(R)$ of isomorphism classes of invertible R-modules. Our first order of business, in this regard, will be to prove the following proposition summarizing the main relation between the two types of R-modules.

Proposition 1.3.4. *Let R be a domain with fraction field F. Suppose that $I \subset F$ is an invertible fractional ideal for R. Then I is invertible as an R-module.*

Conversely, if M is an invertible R-module, then there exists a fractional ideal $I \subset F$ and an R-module isomorphism $M \cong I$.

The proof of Proposition 1.3.4 uses the next three lemmas which may have use outside the statement of the proposition.

Lemma 1.3.5. *Let R be an integral domain with fraction field F. Suppose $I \subset F$ is an invertible fractional ideal for R. Then I is locally free of finite rank.*

Proof. Specifically, we show that if $I \subset F$ is an invertible fractional ideal for R, then I satisfies condition (2) of Theorem 1.1.10 so that I is a finitely generated and projective R-module. From the definition, since I is invertible we have $II^{-1} = R$. This means that there is an expression in F

$$1 = \sum_{i=1}^{n} f_i g_i \qquad f_i \in I,\ g_i \in I^{-1} \text{ for all } 1 \leq i \leq n.$$

Define a map $\phi : R^{\oplus n} \to I$ by sending the standard basis element e_i to f_i. Define a map $\psi : I \to R^{\oplus n}$ by sending $f \in I$ to the element $(fg_1, \ldots, fg_n)$. Both ϕ and ψ are R-module homomorphisms and, for any $f \in I$, we have

$$\phi \circ \psi(f) = \phi((fg_1, ..., fg_n)) = fg_1 f_1 + \cdots + fg_n f_n = f \cdot 1 = f.$$

Thus ψ realizes I as a direct summand of the free R-module $R^{\oplus n}$ as claimed. $\square$

Lemma 1.3.6. *Let R be any commutative ring and let M be a flat R-module. Then the multiplication map*

$$I \otimes_R M \to IM \qquad r \otimes m \mapsto rm$$

is an isomorphism for every ideal $I \subset R$.

Proof. The map $I \otimes_R M \to IM$ is always surjective, regardless if M is flat or not. We show if M is flat then this map is also injective. To prove this, we note that there's a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & I \otimes_R M \to \arrow[r] \arrow[d] & R \otimes_R M \arrow[r] \arrow[d] & R/I \otimes_R M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & IM \arrow[r] & M \arrow[r] & M/IM \arrow[r] & 0
\end{tikzcd}$$

with vertical arrows induced by multiplication maps (sending a pure tensor $r \otimes m$ to the element rm). The middle vertical arrow in this diagram is an isomorphism, hence the left vertical arrow is too. $\square$

Lemma 1.3.7. *Let R be a domain with fraction field $F = R_{(0)}$. Let $I, J \subset F$ be two fractional ideals for R and assume that I is an invertible fractional ideal. Then the canonical surjective map*

$$I \otimes_R J \to IJ \qquad f \otimes g \mapsto fg$$

is an isomorphism of R-modules.

Proof. Since I is an invertible fractional ideal, we have that I is finite rank locally free by Lemma 1.3.5, hence flat as an R-module. Let $f \in R \setminus \{0\}$ be an element so that $fJ \subset R$. Multiplication by f induces an R-module isomorphism $J \cong fJ$. This gives a commutative diagram of R-module homomorphisms

$$\begin{tikzcd} I \otimes_R J \arrow[r] \arrow[d] & IJ \arrow[d] \\ I \otimes_R fJ \arrow[r] & I(fJ) = fIJ \end{tikzcd}$$

where the horizontal arrows are the morphisms gotten from multiplication of simple tensors and the vertical arrows are multiplication by f. Since $fJ \subset R$ is an ideal, the bottom horizontal arrow is an isomorphism by Lemma 1.3.6. Hence the top horizontal arrow is an isomorphism as well. $\square$

Proof of Proposition 1.3.4. The first statement of the proposition follows nearly immediately from the above lemmas. If $I \subset F$ is an invertible fractional ideal for R, then both I and I^{-1} are locally free R-modules of finite rank by Lemma 1.3.5. To check the rank of I, let $\mathfrak{p} \subset R$ be any prime ideal. Then localizing the isomorphism of Lemma 1.3.7 yields an isomorphism

$$I_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} (I^{-1})_{\mathfrak{p}} \cong (I \otimes_R I^{-1})_{\mathfrak{p}} \cong (II^{-1})_{\mathfrak{p}} = R_{\mathfrak{p}}.$$

It follows that $\mathrm{rk}_{R_{\mathfrak{p}}}(I_{\mathfrak{p}}) = 1$. Hence I is invertible as an R-module.

Conversely, if M is an invertible R-module then M is flat. Tensoring the inclusion $R \subset F$ with M gives a series of R-module maps

$$M \cong M \otimes_R R = M \otimes_R F \cong F$$

with the isomorphism $M \otimes_R F = M \otimes_R R_{(0)} \cong R_{(0)} = F$ coming from the fact that R is everywhere rank 1. $\square$

Example 1.3.8. Let k be a field with $-1 \notin k^{\times 2}$, and set $R = k[x,y]/(x^2+y^2-1)$. We saw that the ideal $I = (x, y-1) \subset R$ determined a nonzero element of $\mathrm{Pic}(R)$ in Example 1.2.14. Because of Lemma 1.3.6, there is an isomorphism $I \otimes_R I \cong I^2$ and $I^2 = (x^2, x(y-1), y^2-2y+1) = (y-1)$ is principal. Hence the subgroup generated by $[I]$ inside $\mathrm{Pic}(R)$ is isomorphic with $\mathbb{Z}/2\mathbb{Z}$.

Every fractional ideal for an integral domain R has the structure of an invertible R-module but, not every pair of distinct fractional ideals $I, J \subset F = R_{(0)}$ will have distinct invertible R-module structures. For example, if $(f) \subsetneq R$ is a principal ideal then (f) and R are distinct fractional ideals. However, as R-modules $R \cong (f)$ via the multiplication-by-f map. We can make precise this difference by introducing:

Definition 1.3.9. Let R be an integral domain with field of fractions $F = R_{(0)}$. A fractional ideal $I \subset F$ is a *principal fractional ideal* if I is an R-submodule

$$fR = \{fr : r \in R\} \subset F$$

for some element $f \in F \setminus \{0\}$.

A principal fractional ideal fR is invertible with ideal inverse $(fR)^{-1} = (1/f)R$. We write $I_{pr}(R) \subset I_{fr}(R)$ for the subgroup of all principal fractional ideals.

Theorem 1.3.10. *Let R be an integral domain with field of fractions $F = R_{(0)}$. Then the sequence of group homomorphisms*

$$1 \to R^\times \to F^\times \xrightarrow{f \mapsto fR} I_{fr}(R) \xrightarrow{I \mapsto [I]} \operatorname{Pic}(R) \to 1 \tag{1.3.11}$$

is exact.

Proof. The maps in (1.3.11) are mostly all canonically defined. The two labeled maps are $F^\times \to I_{fr}(R)$, which sends an element f to the principal fractional ideal fR and is readily checked to be a homomorphism, and the map $I_{fr}(R) \to \operatorname{Pic}(R)$, sending a fractional ideal I to the isomorphism class of I as an R-module which is a homomorphism by Lemma 1.3.7. With everything defined, we'll check exactness at each spot.

Clearly the inclusion of units $R^\times \subset F^\times$ is injective. The map $F^\times \to I_{fr}(R)$ has kernel exactly those elements $f \in F^\times$ so that $fR = R$, i.e. those elements $f \in F^\times$ with both $f \in R$ and such that there is some $g \in R$ with $fg = 1$. Hence $f \in R^\times$.

Surjectivity of $I_{fr}(R) \to \operatorname{Pic}(R)$ follows from the converse of Proposition 1.3.4. Lastly, note that there is an isomorphism

$$\operatorname{Hom}_R(R, I) \cong I \qquad \text{defined by } f \mapsto f(1)$$

so that if a fractional ideal $I \in I_{fr}(R)$ is isomorphic with R as an R-module via a map $f : R \to I$, then $I = f(1)R$ and $f(1) \in I \subset F$. $\square$

For a Dedekind domain, the structure of the group $I_{fr}(R)$ is even more explicit. We'll use the following observations to describe the K-theory $K(R)$ of a Dedekind domain completely in Section 1.5 below.

Lemma 1.3.12. *Let R be a Noetherian domain with $F = R_{(0)}$ its field of fractions. Let $I, J \subset F$ be two fractional ideals for R and fix a prime ideal $\mathfrak{p} \subset R$. Then:*

(1) *The localization*

$$I_{\mathfrak{p}} = I \otimes_R R_{\mathfrak{p}} \subset F \otimes_R R_{\mathfrak{p}} = F$$

is a fractional ideal for $R_{\mathfrak{p}}$.

(2) *Localization commutes with fractional ideal products* $(IJ)_{\mathfrak{p}} = I_{\mathfrak{p}}J_{\mathfrak{p}}$.
(3) *Localization commutes with fractional ideal inverses* $(I^{-1})_{\mathfrak{p}} = (I_{\mathfrak{p}})^{-1}$.

Proof. Let $f \in R \setminus \{0\}$ be an element with $fI \subset R$. Then $fI_{\mathfrak{p}} \subset R_{\mathfrak{p}}$ proving (1). The proof of (2) is straightforward. For (3), if $f \in F$ is such that $fI \subset R$ and if $g \in R \setminus \mathfrak{p}$, then

$$(f/g)I \subset (1/g)R \subset R_{\mathfrak{p}}.$$

Hence $(I^{-1})_{\mathfrak{p}} \subset (I_{\mathfrak{p}})^{-1}$. To prove the converse, suppose that I is generated as an R-module by elements $a_1, ..., a_n \in F$. If $f \in F$ is such that $fI_{\mathfrak{p}} \subset R_{\mathfrak{p}}$ then there are equalities

$$fa_1 = \frac{r_1}{s_1}, \cdots, fa_n = \frac{r_n}{s_n}$$

for some elements $r_1, ..., r_n \in R$ and $s_1, ..., s_n \in R \setminus \mathfrak{p}$. Setting $s = s_1 \cdots s_n$, it follows that $(sf)a_i \in R$ for all $i = 1, ..., n$. In other words, $sf \in I^{-1}$ and $f \in (I^{-1})_{\mathfrak{p}}$. □

Lemma 1.3.13. *Let R be a Dedekind domain (i.e. an integrally closed Noetherian domain of Krull dimension* $\mathrm{Kr.dim}(R) = 1$*) and let $F = R_{(0)}$ be its fraction field. If $I \subset F$ is a fractional ideal for R, then I is invertible, i.e. $II^{-1} = R$.*

Proof. Fix a nonzero prime ideal $\mathfrak{p} \subset R$ and note that the local ring $R_{\mathfrak{p}}$ is a DVR. Let π be a uniformizing parameter for $R_{\mathfrak{p}}$ and let v_π be the associated valuation. Let $f \in R$ be such that $fI \subset R$. Then $fI_{\mathfrak{p}} \subset R_{\mathfrak{p}}$ is an ideal of $R_{\mathfrak{p}}$ and so

$$fI_{\mathfrak{p}} = \mathfrak{p}^r = (\pi^r)$$

for some $r \geq 0$. Hence $I_{\mathfrak{p}}$ is generated as an $R_{\mathfrak{p}}$-module by π^{r-s} where $s = v_\pi(f)$. In particular, $\pi^{s-r} \in (I_{\mathfrak{p}})^{-1}$ so that

$$(II^{-1})_{\mathfrak{p}} = I_{\mathfrak{p}}I_{\mathfrak{p}}^{-1} = R_{\mathfrak{p}}$$

because of Lemma 1.3.12. As this is true for each nonzero prime ideal $\mathfrak{p} \subset R$, and hence for every maximal ideal, we must have that the ideal $II^{-1} \subset R$ is contained in no maximal ideal of R, i.e. $II^{-1} = R$. □

Remark 1.3.14. One consequence of the proof of Lemma 1.3.13, which is also interesting, is that if I is a fractional ideal for a discrete valuation ring R with maximal ideal $\mathfrak{m}$ and with uniformizer π, then $I = \pi^r R$ for some integer $r \in \mathbb{Z}$. So, not only is every ideal of R generated by a power of π but, every fractional ideal is also.

Remark 1.3.15. Every fractional ideal I for a Dedekind domain R can uniquely be identified as a product of nonzero prime ideals of R with integer exponents: if $f \in R \setminus \{0\}$ is an element with $fI \subset R$ then there are unique decompositions

$$(f) = \mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_r^{n_r} \quad \text{and} \quad fI = \mathfrak{q}_1^{m_1} \cdots \mathfrak{q}_s^{m_s},$$

as products of prime ideals by [AM69, Corollary 9.4] which allows us to write

$$I = \mathfrak{q}_1^{m_1} \cdots \mathfrak{q}_s^{m_s} \cdot \mathfrak{p}_1^{-n_1} \cdots \mathfrak{p}_r^{-n_r}.$$

Further, this identification is independent of the choice of f: for any nonzero prime ideal $\mathfrak{p} \subset R$ with uniformizing parameter $\pi \in \mathfrak{p}R_\mathfrak{p}$ and associated valuation v_π, the power of $\mathfrak{p}$ that appears in the expansion for I above is determined by the v_π-valuation of any generator for the $R_\mathfrak{p}$-module $I_\mathfrak{p}$.

Therefore, in the case of a Dedekind domain, essentially all of the information contained in the Picard group should already be obtainable from the collection of integral fractional ideals. The following proposition confirms this.

Proposition 1.3.16. *The Picard group* $\mathrm{Pic}(R)$ *of a Dedekind domain* R *can be identified with the set of isomorphism classes of ideals* $J \subset R$.

Proof. Any element of the Picard group $\mathrm{Pic}(R)$ is represented by some fractional ideal and, conversely, all fractional ideals have a corresponding class inside $\mathrm{Pic}(R)$. If I is a fractional ideal for R, then there is an element $f \in R$ and an isomorphism of R-modules

$$I \xrightarrow{\sim} fI = J \subset R \quad \text{defined by } x \mapsto fx.$$

Since the two fractional ideals I and $fI = (fR)I$ represent the same element in $\mathrm{Pic}(R)$, every element of $\mathrm{Pic}(R)$ is represented by an ideal of R.

If $I, J \subset R$ are two ideals which represent the same class in $\mathrm{Pic}(R)$, then by the exactness of (1.3.11) there is an element f in the fraction field $F = R_{(0)}$ so that $(fR)I = J$. If we write $f = g/h$ for two elements $g, h \in R$, then this gives $gI = hJ$. So I and J are isomorphic as R-modules via the composition

$$I \xrightarrow{\cdot g} gI = hJ \xrightarrow{\cdot h^{-1}} J.$$

Conversely, if $I, J \subset R$ are two ideals of R that are isomorphic as R-modules, then let $\varphi : I \to J$ be any such isomorphism. Choose some element $f \in I \setminus \{0\}$. Then for any $x \in I$ we have

$$f\varphi(x) = \varphi(fx) = \varphi(f)x.$$

As $x \in I$ varies, we find that $fJ = \varphi(f)I$ so that I and J represent the same element in $\mathrm{Pic}(R)$. ☐

Weil divisors and the divisor class group

Recall that a prime ideal $\mathfrak{p}$ inside a ring R is said to have *height* n if the supremum, over all chains of prime ideals of R contained in $\mathfrak{p}$, of the lengths of a prime ideal chain $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_r = \mathfrak{p}$ is n. Symbolically,

$$\mathrm{ht}(\mathfrak{p}) = \sup\{r \in \mathbb{Z}_{\geq 0} \mid \text{there exists a chain of prime ideals } \mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_r = \mathfrak{p}\}.$$

A prime ideal $\mathfrak{p} \subset R$ satisfying $\mathrm{ht}(\mathfrak{p}) = 0$ is therefore a minimal prime ideal of R. If R is a domain, then $\mathrm{ht}(\mathfrak{p}) = 0$ implies $\mathfrak{p} = (0)$.

Definition 1.3.17. Let R be a commutative ring. The free abelian group

$$\mathrm{WDiv}(R) = \bigoplus_{\substack{\mathfrak{p} \subset R \text{ prime} \\ \mathrm{ht}(\mathfrak{p})=1}} \mathbb{Z} \cdot \mathfrak{p}$$

indexed by prime ideals $\mathfrak{p} \subset R$ of height $\mathrm{ht}(\mathfrak{p}) = 1$ is called *the group of Weil divisors for* R. An element of $\mathrm{WDiv}(R)$, i.e. a formal linear combination of these prime ideals with integer coefficients, is *a Weil divisor for* R.

A Weil divisor

$$D = \sum_{i \in I} n_i \mathfrak{p}_i$$

is *effective*, written $D \geq 0$, if $n_i \geq 0$ for all $i \in I$. If E is another Weil divisor, then we write $D - E \geq 0$ to mean that $D - E$ is effective. We say that D is *irreducible* if there is a $j \in I$ so that $n_i = 0$ for all $i \in I \setminus \{j\}$ and $n_j = 1$.

Remark 1.3.18. For a Dedekind domain R, each prime ideal $\mathfrak{p} \subset R$ with $\mathrm{ht}(\mathfrak{p}) = 1$ is a maximal ideal. Geometrically, the affine scheme $\mathrm{Spec}(R)$ is one-dimensional, so a curve, and irreducible Weil divisors are in one-to-one correspondence with the closed points of $\mathrm{Spec}(R)$.

For a general ring R, the collection of irreducible Weil divisors is in one-to-one correspondence with closed subschemes of codimension-1 in $\mathrm{Spec}(R)$. An arbitrary Weil divisor for R is then a formal linear combination of these irreducible divisors which may not correspond, in any obvious way, to a subscheme of $\mathrm{Spec}(R)$.

Although I'm not a historian, it seems likely that Weil divisors were introduced in order to analyze the question of whether or not an arbitrary Weil divisor on a given space, like $\mathrm{Spec}(R)$, could be realized as the collection of "zeros and poles of a rational function f counting multiplicities". (Well, historically it was probably more appropriate to consider spaces akin to a complex manifold and, instead of rational functions, one would ask about the possible zeros and poles of complex meromorphic functions as in the Weierstrass factorization theorem).

For any ring R, we can consider an element $f \in R$ as a function on $\mathrm{Spec}(R)$ with value at a point $\mathfrak{p} \in \mathrm{Spec}(R)$ being the class of f in the residue field $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. In this way, we have that f vanishes at a point $\mathfrak{p} \in \mathrm{Spec}(R)$ if and only if $f \in \mathfrak{p}$. If R is a domain, then there are two trivial cases: either $f = 0$, and f vanishes at all $\mathfrak{p} \in \mathrm{Spec}(R)$, or $f \in R^\times$ is a unit and f vanishes at no point $\mathfrak{p} \in \mathrm{Spec}(R)$. The only other possibility is that $f \neq 0$ is a nonunit, in which case the ideal (f) is contained in at least one maximal ideal $\mathfrak{m} \subset R$.

If R is a Noetherian integral domain, then the vanishing set of f is made up of a collection of finitely many irreducible components corresponding to the finitely many prime ideals of R minimally containing the ideal (f). The following theorem, known as Krull's principal ideal theorem (or Krull's Hauptidealsatz), implies that in this case each minimal prime ideal for (f) has $\mathrm{ht}(\mathfrak{p}) = 1$.

Theorem 1.3.19 (Krull's Principal Ideal Theorem)**.** *Let R be a Noetherian ring. Let $(f) \subsetneq R$ be a proper, principal ideal of R. Then for each minimal prime ideal $\mathfrak{p}$ of (f), there is an inequality* $\mathrm{ht}(\mathfrak{p}) \leq 1$.

Reference. This is proved in most texts on commutative algebra, see for instance [Kun85, Theorem 3.1] for a nonstandard reference. One can also find a proof of this result in the more standard [AM69, Corollary 11.17], in [Eis95, Theorem 10.1], and online at [Sta19, Tag 00KV]. □

All of this is to say that a Weil divisor really is the appropriate geometric object capturing the vanishing of a nonzero, nonunit function $f \in R$. However, there's still the problem of accurately capturing the notion of "the order of vanishing" of such a function f. If we assume that R is a UFD, then we could just write $f = u\pi_1^{r_1} \cdots \pi_s^{r_s}$ uniquely as a product of a unit $u \in R^\times$ and some prime elements $\pi_1, ..., \pi_s \in R$ to some powers $r_1, ..., r_s \in \mathbb{N}$ and then define the order of vanishing of f at a prime ideal $(\pi) \in \mathrm{Spec}(R)$ to be r_i if $\pi = \pi_i$ for some $1 \leq i \leq s$ and 0 otherwise. This works well in simple cases, e.g. if $R = k[x]$ is a polynomial ring in one variable over a field k and f is an element like $f = (x-1)^2(x+1)$, but most rings are not UFD's.

The next best assumption we could work with is that R is a Dedekind domain. Then R may not be a UFD but, the ideal $(f) \subsetneq R$ still admits a unique factorization into a product of prime ideals $(f) = \mathfrak{p}_1^{r_1} \cdots \mathfrak{p}_s^{r_s}$ with $r_1, ..., r_s \in \mathbb{N}$ so that we could define the order of vanishing of f at a prime ideal $\mathfrak{p}$ to be r_i if $\mathfrak{p} = \mathfrak{p}_i$ for some $1 \leq i \leq s$ and 0 otherwise. Of course, the number r_i can, in this case, also be determined using only information coming from the ring $R_{\mathfrak{p}_i}$ by writing $f = u\pi_i^{r_i}$ for some unit $u \in R_{\mathfrak{p}_i}^\times$ and for some uniformizer π_i of the DVR $R_{\mathfrak{p}_i}$.

This last point requires us to assume much less than for R to be a Dedekind domain. Specifically, to define the order of vanishing of the function f at a prime ideal $\mathfrak{p}$ of height $\mathrm{ht}(\mathfrak{p}) = 1$ we could start by assuming that the localization $R_{\mathfrak{p}}$ is a DVR, picking a uniformizer π for $R_{\mathfrak{p}}$ with associated valuation v_π, and taking the order as the number $v_\pi(f)$; this recovers our first two definitions, in both the case of a UFD and the case of a Dedekind domain, at the very least.

A sufficient condition for the localization $R_{\mathfrak{p}}$ to be a DVR at each prime ideal $\mathfrak{p}$ with $\operatorname{ht}(\mathfrak{p}) = 1$ is the assumption that R is an integrally closed Noetherian domain. Then for any prime ideal $\mathfrak{p} \subset R$ the localization $R_{\mathfrak{p}}$ is also integrally closed [AM69, Proposition 5.13] and, for primes $\mathfrak{p}$ with $\operatorname{ht}(\mathfrak{p}) = 1$, this implies that $R_{\mathfrak{p}}$ is a DVR [AM69, Proposition 9.2]. In fact, if we're willing to work with Noetherian domains, then assuming R is integrally closed is nearly also necessary for defining the order of vanishing for a function $f \in R$ in this way because of the following theorem.

Theorem 1.3.20. *Let R be a Noetherian integral domain with fraction field F. Then R is integrally closed in F if and only if the following conditions are satisfied:*

(1) *for all prime ideals $\mathfrak{p} \subset R$ with $\operatorname{ht}(\mathfrak{p}) = 1$, the ring $R_{\mathfrak{p}}$ is a DVR;*

(2) *there is an equality*

$$R = \bigcap_{\substack{\mathfrak{p} \subset R \text{ prime} \\ \operatorname{ht}(\mathfrak{p})=1}} R_{\mathfrak{p}}$$

inside the field F.

Reference. Assume that R is integrally closed. Then we've observed already that $R_{\mathfrak{p}}$ is a DVR for all prime ideals $\mathfrak{p} \subset R$ with $\operatorname{ht}(\mathfrak{p}) = 1$ by [AM69, Proposition 9.2]. Moreover, the subring $R \subset F$ is equal to the intersection given in (2), see [Rei95, §8.10, Theorem] or [Mat89, Theorem 11.5].

Conversely, if (1) and (2) hold then R is integrally closed as it is the intersection of valuation rings, see [AM69, Corollary 5.22]. □

With that out of the way, it's finally time to make the following definition.

Definition 1.3.21. Assume that R is an integrally closed Noetherian domain with fraction field $F = R_{(0)}$. Let $\mathfrak{p} \subset R$ be a prime ideal with height $\operatorname{ht}(\mathfrak{p}) = 1$. Choose a uniformizer $\pi \in \mathfrak{p}R_{\mathfrak{p}}$ and let $v_\pi : F^\times \to \mathbb{Z}$ be the valuation of F induced by π. We define *the order of vanishing of $f \in F^\times$ at $\mathfrak{p}$* as the integer $\operatorname{ord}_{\mathfrak{p}}(f) := v_\pi(f)$.

Remark 1.3.22. Keep the set-up $(R, F, \mathfrak{p}, \pi, f)$ of the above definition. The order of vanishing $\operatorname{ord}_{\mathfrak{p}}(f)$ can usually be computed in-practice as follows.

Since F is also the fraction field of $R_{\mathfrak{p}}$, the element $f \in F^\times$ can be written as a ratio $f = g/h$ for two elements $g, h \in R_{\mathfrak{p}}$. One can then write $g = u\pi^r$ and $h = v\pi^s$ for units $u, v \in R_{\mathfrak{p}}^\times$ and for integers $r, s > 0$. Then $v_\pi(f) = v_\pi(g) - v_\pi(h) = r - s$. This number is the same regardless of the choice of uniformizer $\pi \in \mathfrak{p}R_{\mathfrak{p}}$, since the valuation itself is independent of π, and:

(1) $v_\pi(f) \geq 0$ if and only if $f \in R_{\mathfrak{p}}$

(2) $v_\pi(f) > 0$ if and only $f \in \mathfrak{p}R_{\mathfrak{p}}$.

Lemma 1.3.23. *Let R be an integrally closed Noetherian domain with $F = R_{(0)}$. Let* $\mathrm{div} : F^\times \to \mathrm{WDiv}(R)$ *be the map defined by*

$$\mathrm{div}(f) = \sum_{\substack{\mathfrak{p}\subset R \text{ prime}\\ \mathrm{ht}(\mathfrak{p})=1}} \mathrm{ord}_{\mathfrak{p}}(f)\cdot \mathfrak{p}.$$

Then div *is a well-defined group homomorphism.*

Proof. For any prime ideal $\mathfrak{p} \subset R$ with $\mathrm{ht}(\mathfrak{p}) = 1$, the valuation v_π induced by a uniformizer π of $\mathfrak{p}$ yields a group homomorphism $v_\pi : F^\times \to \mathbb{Z}$ which is identically the component $\mathrm{ord}_{\mathfrak{p}} : F^\times \to \mathbb{Z}\cdot\mathfrak{p}$ of div. Together the various $\mathrm{ord}_{\mathfrak{p}}$ maps give a well-defined homomorphism to the product

$$\mathrm{div}' : F^\times \to \prod_{\substack{\mathfrak{p}\subset R \text{ prime}\\ \mathrm{ht}(\mathfrak{p})=1}} \mathbb{Z}\cdot\mathfrak{p} \quad \text{defined by } \mathrm{div}'(f) = (\mathrm{ord}_{\mathfrak{p}}(f))_{\mathfrak{p}}.$$

Now the map div' has image in the subgroup $\mathrm{WDiv}(R)$ since for any $f \in F^\times$ we have $\mathrm{ord}_{\mathfrak{p}}(f) = 0$ for all but finitely many prime ideals $\mathfrak{p}$ of $\mathrm{ht}(\mathfrak{p}) = 1$ (if we write $f = g/h$ for $g, h \in R$, then both elements g and h are contained in only finitely many height one primes since R is Noetherian). The map $\mathrm{div} : F^\times \to \mathrm{WDiv}(R)$ is then the induced homomorphism with this restricted target. □

Definition 1.3.24. Let R be an integrally closed Noetherian domain with field of fractions $F = R_{(0)}$. We call a Weil divisor $D \in \mathrm{WDiv}(R)$ a *principal Weil divisor* if D is in the image of the divisor map div, i.e. if there is a rational function $f \in F^\times$ so that $D = \mathrm{div}(f)$.

The cokernel of the divisor map, i.e. the quotient of the group of all Weil divisors by the subgroup consisting of all principal Weil divisors, is called *the divisor class group of* R and written as $\mathrm{Cl}(R) = \mathrm{WDiv}(R)/\mathrm{div}(R)$.

By analogy with Theorem 1.3.10, there is a corresponding exact sequence.

Proposition 1.3.25. *Let R be an integrally closed Noetherian domain with field of fractions F. Then the sequence of group homomorphisms*

$$1 \to R^\times \to F^\times \xrightarrow{\mathrm{div}} \mathrm{WDiv}(R) \to \mathrm{Cl}(R) \to 0 \tag{1.3.26}$$

is exact.

Proof. The only thing to check is that the kernel of $\mathrm{div} : F^\times \to \mathrm{WDiv}(R)$ is the group of units $R^\times$ of R. So let $f \in F^\times$ be such that $\mathrm{ord}_{\mathfrak{p}}(f) = 0$ for all $\mathfrak{p} \subset R$ prime with $\mathrm{ht}(\mathfrak{p}) = 1$. This implies, in particular, that the valuation of f corresponding to any such prime also vanishes; hence $f \in R_{\mathfrak{p}} \setminus \mathfrak{p}R_{\mathfrak{p}}$ for all such primes. By (2) of

Theorem 1.3.20, it follows that $f \in R$ is an element such that (f) is contained in no height one prime ideal.

If f were a nonunit, then the ideal (f) would be contained in some maximal ideal $\mathfrak{m}$ and hence would be contained in some prime ideal $\mathfrak{p}$ which was minimal with respect to containing (f). But by Theorem 1.3.19, the height of $\mathfrak{p}$ must be one and this would imply $f \in \mathfrak{p}R_{\mathfrak{p}}$. So f must be a unit (i.e. $f \in R^{\times}$). $\square$

The similarity between the sequence (1.3.26) of Proposition 1.3.25 and (1.3.11) of Theorem 1.3.10 is overly suggestive. To compare the two directly, we'll construct a natural map

$$\mathrm{div} : I_{fr}(R) \to \mathrm{WDiv}(R)$$

so that for rational function $f \in F^{\times}$ in the fraction field of R there is an equality $\mathrm{div}(fR) = \mathrm{div}(f)$ for the principal fractional ideal fR of R.

If I is an invertible fractional ideal for an integrally closed Noetherian domain R, and if $\mathfrak{p} \subset R$ is a prime ideal of height $\mathrm{ht}(\mathfrak{p}) = 1$, then $R_{\mathfrak{p}}$ is a DVR. Suppose that the maximal ideal $\mathfrak{p}R_{\mathfrak{p}}$ is generated by π. Then by Remark 1.3.14 the fractional ideal $I_{\mathfrak{p}}$ for $R_{\mathfrak{p}}$ is equal to $\pi^r R_{\mathfrak{p}}$ for some $r \in \mathbb{Z}$. Define $\mathrm{ord}_{\mathfrak{p}}(I) = r$.

Theorem 1.3.27. *Let R be an integrally closed Noetherian domain with $F = R_{(0)}$. Then $\mathrm{div} : I_{fr}(R) \to \mathrm{WDiv}(R)$, defined on an invertible fractional ideal I of R by*

$$\mathrm{div}(I) = \sum_{\substack{\mathfrak{p} \subset R \text{ prime} \\ \mathrm{ht}(\mathfrak{p})=1}} \mathrm{ord}_{\mathfrak{p}}(I) \cdot \mathfrak{p},$$

is a well-defined group homomorphism with $\mathrm{div}(fR) = \mathrm{div}(f)$ for any $f \in F^{\times}$. Hence, there is a commutative diagram with exact rows

$$\begin{tikzcd}
1 \arrow[r] & R^{\times} \arrow[r] \arrow[d, equal] & F^{\times} \arrow[r] \arrow[d, equal] & I_{fr}(R) \arrow[r] \arrow[d, "\mathrm{div}"] & \mathrm{Pic}(R) \arrow[r] \arrow[d, "c_1"] & 1 \\
1 \arrow[r] & R^{\times} \arrow[r] & F^{\times} \arrow[r, "\mathrm{div}"] & \mathrm{WDiv}(R) \arrow[r] & \mathrm{Cl}(R) \arrow[r] & 0
\end{tikzcd}$$

where $c_1 : \mathrm{Pic}(R) \to \mathrm{Cl}(R)$ is the induced map on quotients. Moreover, both of the following statements are true:

(1) *the morphism c_1 is injective,*

(2) *c_1 is surjective if and only if $R_{\mathfrak{p}}$ is a UFD for every prime ideal $\mathfrak{p} \subset R$.*

Proof. There's a lot to prove, so we'll work our way down the theorem statement. First, we show that $\mathrm{div} : I_{fr}(R) \to \mathrm{WDiv}(R)$ is a well-defined homomorphism. Since I is finitely generated (see Remark 1.3.2), it follows that $\mathrm{ord}_{\mathfrak{p}}(I) = 0$ for all but finitely many prime ideals $\mathfrak{p} \subset R$ with $\mathrm{ht}(\mathfrak{p}) = 1$. Hence div is well-defined. That div is a group homomorphism can then be seen using Lemma 1.3.12.

Compatibility between the two div maps follows immediately, and hence the given diagram is commutative. We next show that c_1 is injective and, to do this, we actually show that div : $I_{fr}(R) \to \mathrm{WDiv}(R)$ is injective. So, suppose that I is an invertible fractional ideal with $\mathrm{div}(I) = 0$. This implies that $I \subset R$ is an integral fractional ideal of R (so, really, an ideal) since

$$I \subset \bigcap_{\substack{\mathfrak{p}\subset R \text{ prime} \\ \mathrm{ht}(\mathfrak{p})=1}} I_{\mathfrak{p}} = \bigcap_{\substack{\mathfrak{p}\subset R \text{ prime} \\ \mathrm{ht}(\mathfrak{p})=1}} R_{\mathfrak{p}} = R$$

with the last equality coming from (2) of Theorem 1.3.20. By the same reasoning $I^{-1} \subset R$ is also an ideal. Now from the containments

$$R = II^{-1} \subset IR \subset I \subset R,$$

we see that $I = R$.

Lastly, we need to show that c_1 is surjective if and only if $R_{\mathfrak{p}}$ is a UFD for all prime ideals $\mathfrak{p} \subset R$. The converse first: if $\mathfrak{p} \subset R$ is a prime ideal with $\mathrm{ht}(\mathfrak{p}) = 1$, then $\mathfrak{p}$ is finitely generated and, for every prime ideal $\mathfrak{q} \subset R$, the localization $\mathfrak{p}_{\mathfrak{q}} = \mathfrak{p}R_{\mathfrak{q}} \subset R_{\mathfrak{q}}$ is isomorphic with $R_{\mathfrak{q}}$ since $R_{\mathfrak{q}}$ is a UFD; hence $\mathfrak{p}$ is invertible as an R-module. Now we still need to show that $\mathfrak{p}$ is also an invertible fractional ideal. By Proposition 1.3.4, there is an invertible fractional ideal $\mathfrak{h}$ which is isomorphic as an R-module with $\mathrm{Hom}_R(\mathfrak{p}, R)$. By Lemmas 1.3.7 and 1.2.8 there are isomorphisms

$$\mathfrak{p}\mathfrak{h} \cong \mathfrak{p} \otimes_R \mathfrak{h} \cong \mathfrak{p} \otimes_R \mathrm{Hom}_R(\mathfrak{p}, R) \cong R.$$

This means that the fractional ideal $\mathfrak{p}\mathfrak{h}$ is principal, so there is an $f \in F^{\times}$ with $fR = \mathfrak{p}\mathfrak{h}$. Since $\mathfrak{p} = (fR)\mathfrak{h}^{-1}$ is a product of two invertible fractional ideals, it follows that $\mathfrak{p}$ is an invertible fractional ideal. Of course, $\mathrm{div}(\mathfrak{p}) = \mathfrak{p}$ so that div is a surjection; it follows from this that c_1 is surjective.

For the forward direction, assume that c_1 is surjective. We need to show that $R_{\mathfrak{p}}$ is a UFD for all prime ideals $\mathfrak{p} \subset R$. The argument uses the following lemma.

Lemma 1.3.28. *Let R be a Noetherian domain. Then R is a UFD if and only if each prime ideal $\mathfrak{p} \subset R$ with $\mathrm{ht}(\mathfrak{p}) = 1$ is a principal ideal.*

Proof. Let R be a UFD. Pick a height one prime ideal $\mathfrak{p} \subset R$ and a set of generators $\mathfrak{p} = (f_1, ..., f_n)$ with $f_i \neq 0$ for all $1 \leq i \leq n$. If $f_1 = \pi_1 \cdots \pi_r$ is a factorization into irreducible elements, then $\pi_i \in \mathfrak{p}$ for some $i \in \{1, ..., r\}$. So there is a containment $0 \subsetneq (\pi_i) \subset \mathfrak{p}$. As R is a UFD, the ideal (π_i) is prime and, since $\mathfrak{p}$ has height one, we must have $(\pi_i) = \mathfrak{p}$.

Conversely, assume that all prime ideals of R with height one are principal. In this case, all irreducilbe elements are prime elements. Indeed, if π is an irreducible element of R then for any minimal prime ideal $\mathfrak{p}$ containing (π), we have $\mathrm{ht}(\mathfrak{p}) = 1$ by Theorem 1.3.19. But then $\mathfrak{p} = (x)$ is principal by assumption, so $\pi = xy$ for some $y \in R$. As π is irreducible, it follows that y is a unit, hence $(\pi) = \mathfrak{p}$.

If $f \in R$ is a nonzero, nonunit element then there is a factorization of f into irreducible elements of R since R is Noetherian. Suppose that we have two such factorizations

$$f = \pi_1 \cdots \pi_r \quad \text{and} \quad f = t_1 \cdots t_s$$

where the elements π_i for $1 \le i \le r$ are irreducible as are the elements t_j for $1 \le j \le s$. Since π_1 is prime, there is some $j \in \{1, ..., s\}$ with $t_j \in (\pi_1)$. If we write $t_j = u_1\pi_1$ for some unit $u_1 \in R$ then, after substituting, we have

$$f = \pi_1 \cdots \pi_r \quad \text{and} \quad f = u_1 t_1 \cdots t_{j-1}\pi_1 t_{j+1} \cdots t_s$$

so that $\pi_2 \cdots \pi_r = u_1 t_1 \cdots t_{j-1} t_{j+1} \cdots t_s$ as R is a domain. Continuing in this way, we eventually find that $\pi_r = u t_{i_1} \cdots t_{i_k}$ for some $i_1, ..., i_k \in \{1, ..., s\}$ and for a unit $u = u_1 \cdots u_{r-1} \in R$. If $k > 1$, then it would follow that one of the elements t_i for $i \in \{1, ..., s\}$ was a unit, which isn't true. Hence $k = 1$ so that $r = s$ and it follows that R has unique factorization. $\square$

Coming back to the proof of Theorem 1.3.27, the lemma says that in order to show that $R_\mathfrak{p}$ is a UFD for any given prime ideal $\mathfrak{p} \subset R$, it suffices to check that each height one prime ideal of $R_\mathfrak{p}$ is principal. Let $\mathfrak{q} \subset R$ be a prime ideal with $\mathfrak{q}R_\mathfrak{p}$ a height one prime ideal of $R_\mathfrak{p}$. Then $\mathfrak{q}$ must also have $\mathrm{ht}(\mathfrak{q}) = 1$ inside R.

Since c_1 is surjective, so is $\mathrm{div} : I_{fr}(R) \to \mathrm{WDiv}(R)$. This means that there is an invertible fractional ideal I for R with $\mathrm{div}(I) = \mathfrak{q}$. Localizing, we see that the map $\mathrm{div} : I_{fr}(R_\mathfrak{p}) \to \mathrm{WDiv}(R_\mathfrak{p})$ is defined so that $\mathrm{div}(I_\mathfrak{p}) = \mathfrak{q}$. But, over $R_\mathfrak{p}$, the fractional ideal $I_\mathfrak{p}$ is free and, therefore, trivial inside of $\mathrm{Pic}(R_\mathfrak{p})$. By the exactness of Theorem 1.3.10, this means that $I_\mathfrak{p} = fR_\mathfrak{p}$ for some $f \in F^\times$. Since this implies $\mathrm{ord}_\mathfrak{r}(f) \ge 0$ for all prime ideals $\mathfrak{r} \subset R_\mathfrak{p}$ of height $\mathrm{ht}(\mathfrak{r}) = 1$, we have that $f \in R_\mathfrak{p}$ by (2) of Theorem 1.3.20.

We claim that $\mathfrak{q}R_\mathfrak{p}$ is the principal ideal generated by f. Indeed, if $g \in \mathfrak{q}R_\mathfrak{p}$ is any nonzero element then

$$\mathrm{ord}_\mathfrak{r}(g/f) = \mathrm{ord}_\mathfrak{r}(g) - \mathrm{ord}_\mathfrak{r}(f) \ge 0$$

for all primes $\mathfrak{r} \subset R_\mathfrak{p}$ of height $\mathrm{ht}(\mathfrak{r}) = 1$ so that $g/f \in R_\mathfrak{p}$ by another application of part (2) from Theorem 1.3.20. Hence $g = (g/f)f$ and $(f) = \mathfrak{q}R_\mathfrak{p}$. Since $\mathfrak{q}R_\mathfrak{p}$ was arbitrary, Lemma 1.3.28 now shows that $R_\mathfrak{p}$ is a UFD as claimed. $\square$

Example 1.3.29. If R is a Dedekind domain, then we can identify the map $\mathrm{div} : I_{fr}(R) \to \mathrm{WDiv}(R)$ with the canonical morphism sending a fractional ideal I with factorization

$$I = \mathfrak{p}_1^{r_1} \cdots \mathfrak{p}_s^{r_s}, \qquad r_1, ..., r_s \in \mathbb{Z}$$

to the Weil divisor $D = r_1\mathfrak{p}_1 + \cdots + r_s\mathfrak{p}_s$. Compare with Remark 1.3.15.

Remark 1.3.30. The map $c_1 : \mathrm{Pic}(R) \to \mathrm{Cl}(R)$ of Theorem 1.3.27 composed with the determinant from Corollary 1.2.13 yields a morphism (also named c_1 by abuse of notation)

$$c_1 : K(R) \xrightarrow{\det} \mathrm{Pic}(R) \to \mathrm{Cl}(R)$$

called *the first Chern class homomorphism*. Given a finitely generated projective R-module P, the element $c_1([P]) \in \mathrm{Cl}(R)$ is called *the first Chern class of* P.

For a given Noetherian domain R, the assumption that the localization $R_\mathfrak{p}$ is a UFD for each prime ideal $\mathfrak{p} \subset R$ may sound like a surprisingly strong constraint. If $R_\mathfrak{p}$ is a UFD for every prime ideal $\mathfrak{p} \subset R$, then since a UFD is integrally closed, and since being integrally closed is a local property [AM69, Proposition 5.13], it follows that a domain R with this property is integrally closed.

From the other direction, if R is a UFD, then for any prime ideal $\mathfrak{p} \subset R$ the localization $R_\mathfrak{p}$ is a UFD. So any UFD gives an example of a ring with this property. But are these the only examples? To make the discussion easier, we introduce the definition:

Definition 1.3.31. A ring R is said to be *locally factorial* if the localization $R_\mathfrak{p}$ is a unique factorization domain for each prime ideal $\mathfrak{p} \subset R$.

Example 1.3.32. If R is a Dedekind domain, then R is locally factorial since for any prime ideal $\mathfrak{p} \subset R$, the localization $R_\mathfrak{p}$ is either a field or a DVR (and a DVR, being a PID, is a UFD).

One might know already that there are Dedekind domains which do not have unique factorization (we'll see some soon), so there are plenty of rings which *are* locally factorial and which *are not* unique factorization domains. In Section 1.5, we'll see that there exists a natural class of rings, with a very appealing geometric interpretation, which will provide us with many more examples of locally factorial rings. For now, though, we prove the following corollary to Theorem 1.3.27 which will help us in analyzing examples.

Corollary 1.3.33. *Let R be an integrally closed Noetherian domain. Then R is a UFD if and only if* $\mathrm{Cl}(R) = 0$.

Proof. By Lemma 1.3.28, it suffices to show that $\mathrm{Cl}(R) = 0$ if and only if each prime ideal $\mathfrak{p} \subset R$ with $\mathrm{ht}(\mathfrak{p}) = 1$ is a principal ideal. For the forward direction, if $\mathrm{Cl}(R) = 0$ then each irreducible Weil divisor $\mathfrak{p}$ is a principal divisor, so there is an element $f \in F = R_{(0)}$ such that $\mathrm{div}(f) = \mathfrak{p}$. But, as in last paragraph of the proof of Theorem 1.3.27, this implies that $\mathfrak{p} = (f)$.

Conversely, if each prime ideal $\mathfrak{p} \subset R$ with $\mathrm{ht}(\mathfrak{p}) = 1$ is a principal ideal, then choose one such ideal, say $\mathfrak{q} \subset R$, and let $\mathfrak{q} = (\pi)$ be generated by some $\pi \in R$. We claim that $\mathrm{div}(\pi) = \mathfrak{q}$. Clearly $\mathrm{ord}_{\mathfrak{q}}(\pi) = 1$; we need to show that $\mathrm{ord}_{\mathfrak{p}}(\pi) = 0$ for all other height one prime ideals $\mathfrak{p} \neq \mathfrak{q}$. If $\mathfrak{r}$ was such an ideal, generated by some element $\xi \in R$, then $\mathrm{ord}_{\mathfrak{r}}(\pi) = a$ if and only $\pi = u\xi^a$ for some unit $u \in R$. If $a \geq 1$, then since π is a prime element, we would necessarily have π divides ξ. Thus we obtain the contradiction $(0) \subsetneq \mathfrak{r} \subsetneq \mathfrak{q}$ since $\mathrm{ht}(\mathfrak{q}) = 1$. $\square$

Remark 1.3.34. If R is a Dedekind domain, then $\mathrm{Cl}(R) = 0$ implies $\mathrm{Pic}(R) = 0$. By Proposition 1.3.16 the vanishing $\mathrm{Pic}(R) = 0$ then implies that R is a PID. Hence, for a Dedekind domain R, we have $\mathrm{Cl}(R) = 0$ if and only if R is a PID.

Example 1.3.35. Let k be a field where -1 is not a square of any element of k. Then the ring $R = k[x, y]/(x^2 + y^2 - 1)$ is a Dedekind domain (see Exercise 1.3.4). Since $\mathrm{Pic}(R) \neq 0$ by Example 1.2.14, we have from Theorem 1.3.27 that R is a locally factorial ring which is not a UFD.

Example 1.3.36. Let k be a field where $-1 \notin k^{\times 2}$ as in the above example. Let $R = k[x, y, z]/(xy - z^2 + 1)$ as in Example 1.2.16. By Exercise 1.3.5, the ring R is integrally closed. Since $\mathrm{Pic}(R) \neq 0$, the ring R is not a UFD. Here R is still locally factorial, as one can check in Exercise 1.3.7, but R is not a Dedekind domain since the Krull dimension of R is 2.

Example 1.3.37 (Compare with [Har77, Ch. 2, Example 6.5.2])**.** Let k be any field, and let $R = k[x, y, z]/(xy - z^2)$. We will show $\mathrm{Pic}(R) = 0$ and $\mathrm{Cl}(R) \cong \mathbb{Z}/2\mathbb{Z}$. One can check that R is a Noetherian domain (using Eisenstein's criterion) and by Exercise 1.3.3 the ring R is integrally closed. So this provides at least one example of an integrally closed Noetherian domain which is not locally factorial.

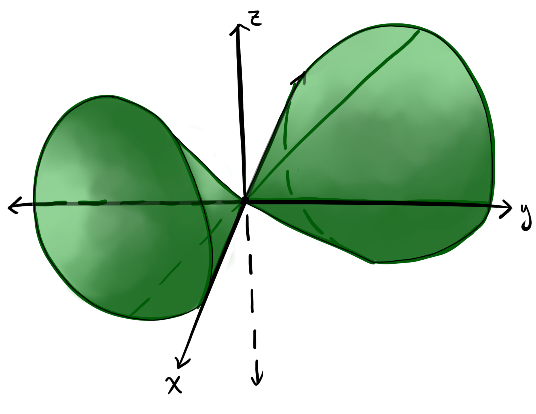

Figure 1.3: The vanishing locus $V(xy - z^2)$ inside $\mathbb{A}^3$

We start by observing the element $y \in R$ has the property that

$$R/yR \cong k[x,z]/(z^2) \quad \text{and} \quad R_y \cong k[y, y^{-1}, z].$$

The only minimal prime ideal containing $(y) \subset R$ is then of the form (y, z) which corresponds to the nilradical of R/yR.

Now there is a natural way to compare the exact sequences from (1.3.26) applied to R and R_y, which ends up as a commutative ladder like so:

$$\begin{tikzcd}
1 \arrow[r] & R^\times \arrow[r] \arrow[d] & F^\times \arrow[r, "\mathrm{div}"] \arrow[d, equal] & \mathrm{WDiv}(R) \arrow[r] \arrow[d] & \mathrm{Cl}(R) \arrow[r] \arrow[d] & 0 \\
1 \arrow[r] & R_y^\times \arrow[r] & F^\times \arrow[r, "\mathrm{div}"] & \mathrm{WDiv}(R_y) \arrow[r] & \mathrm{Cl}(R_y) \arrow[r] & 0.
\end{tikzcd}$$

Here the map $\mathrm{WDiv}(R) \to \mathrm{WDiv}(R_y)$ is the canonical projection with kernel the subgroup $\mathbb{Z}\mathfrak{p}$ where $\mathfrak{p} = (y, z)$ is the minimal prime ideal containing y. Since the map $F^\times/R^\times \to F^\times/R_y^\times$ is a surjection, the subgroup $\mathbb{Z}\mathfrak{p}$ surjects onto the kernel of the induced map $\mathrm{Cl}(R) \to \mathrm{Cl}(R_y)$. But, R_y is a UFD, so by Corollary 1.3.33 $\mathrm{Cl}(R_y) = 0$ and $\mathrm{Cl}(R)$ is then generated by the class $[\mathfrak{p}]$.

For any height one prime ideal $\mathfrak{q} \subset R$ with $\mathfrak{q} \neq \mathfrak{p}$, we must have $\mathrm{ord}_\mathfrak{q}(y) = 0$ since $y \notin \mathfrak{q}$. A computation shows that $\mathrm{ord}_\mathfrak{p}(y) = 2$ since $R_{(y,z)}$ is generated by z and in this ring $y = z^2/x$. Hence $\mathrm{div}(y) = 2\mathfrak{p}$ so that, by definition, there is a relation $2[\mathfrak{p}] = 0$ inside $\mathrm{Cl}(R)$. We still need to show that $[\mathfrak{p}] \neq 0$ inside $\mathrm{Cl}(R)$.

There are two possibilities at this point: either $\mathrm{Cl}(R) = 0$ or $\mathrm{Cl}(R) \cong \mathbb{Z}/2\mathbb{Z}$. If $\mathrm{Cl}(R) = 0$, then because of Theorem 1.3.27 the ring R would be locally factorial. We're going to show that this is not the case by exhibiting a maximal ideal $\mathfrak{m} \subset R$ and a height one prime ideal $\mathfrak{p} \subset R$ with the property that $\mathfrak{p}R_\mathfrak{m}$ is not principal inside $R_\mathfrak{m}$. By Lemma 1.3.28, this implies $R_\mathfrak{m}$ is not a UFD and therefore we must have $\mathrm{Cl}(R) \cong \mathbb{Z}/2\mathbb{Z}$. Since we're showing that R is not locally factorial, this also implies that $\mathrm{Pic}(R) = 0$.

We take for $\mathfrak{m}$ the ideal $\mathfrak{m} = (x, y, z)$ and for $\mathfrak{p}$ we take $\mathfrak{p} = (y, z)$. Clearly $\mathfrak{p}R_\mathfrak{m} \subset \mathfrak{m}R_\mathfrak{m}$. The quotient $\mathfrak{m}R_\mathfrak{m}/(\mathfrak{m}R_\mathfrak{m})^2 \cong \mathfrak{m}/\mathfrak{m}^2$ is an $R/\mathfrak{m} \cong k$-vector space of dimension 3 (spanned linearly by the elements x, y, z). If $\mathfrak{p}R_\mathfrak{m}$ was principal, then the image of $\mathfrak{p}$ in $\mathfrak{m}/\mathfrak{m}^2$ would be a 1-dimensional k-vector subspace but, this is not the case since the image contains both x and y.

Exercises for Section 1.3

1. (*Cartier divisors*). Let R be an integral domain and $F = R_{(0)}$ its fraction field. Write $\overline{\mathrm{CDiv}}(R)$ for the set of all sets of pairs $\{(s_i, f_i)\}_{i \in I}$ consisting of pairs of elements $s_i \in R$ and elements $f_i \in F$ satisfying the properties:
   (1) the basic opens $D(s_i) \subset \mathrm{Spec}(R)$ cover $\mathrm{Spec}(R)$ as $i \in I$ varies;

(2) for all $i, j \in I$ there is a unit $\phi_{ij} \in R_{s_i s_j}^{\times}$ so that $f_j = \phi_{ij} f_i$;
(3) for any triple $i, j, k \in I$ we have $\phi_{jk}\phi_{ij} = \phi_{ik}$ inside $R_{s_i s_j s_k}^{\times}$.

We define two such sets $\{(s_i, f_i)\}_{i\in I}$ and $\{(t_j, g_j)\}_{j\in J}$ to be equivalent if for all pairs $(i, j) \in I \times J$ there is some $\rho_{ij} \in R_{s_i t_j}^{\times}$ so that $f_i = \rho_{ij} g_j$. The collection of these sets, up to this equivalence, is denoted $\mathrm{CDiv}(R)$ and an arbitrary element of $\mathrm{CDiv}(R)$ is called *a Cartier divisor for R*.

(a) Let $\{(s_i, f_i)\}_{i\in I}$ represent a Cartier divisor D for R and let E be another Cartier divisor represented by $\{(t_j, g_j)\}_{j\in J}$. Define $D + E$ as the Cartier divisor represented by $\{(s_i t_j, f_i g_j)\}_{(i,j)\in I\times J}$. Show that the set $\mathrm{CDiv}(R)$ is naturally an abelian group with this operation.
(b) Let D be a Cartier divisor represented by a collection of pairs $\{(s_i, f_i)\}_{i\in I}$. Associate to D the R-submodule $I(D) \subset F$ defined as

$$I(D) = \bigcap_{i\in I} f_i R_{s_i}.$$

Show that $I(D)$ is an invertible fractional ideal for R.
(c) Define a map $\psi : \mathrm{CDiv}(R) \to I_{fr}(R)$ by sending a Cartier divisor D to the fractional ideal $I(D)$. Show that ψ is an isomorphism of groups.
(d) Prove that the composition of the homomorphisms $\psi : \mathrm{CDiv}(R) \to I_{fr}(R)$ and $\mathrm{div} : I_{fr}(R) \to \mathrm{WDiv}(R)$ has the following interpretation. If a Cartier divisor D is represented by $\{(s_i, f_i)\}_{i\in I}$, then

$$\mathrm{div} \circ \psi(D) = \sum_{\substack{\mathfrak{p}\subset R \text{ prime} \\ \mathrm{ht}(\mathfrak{p})=1}} \mathrm{ord}_{\mathfrak{p}}(f_i)\mathfrak{p} \quad \text{for any } i \text{ with } s_i \notin \mathfrak{p}.$$

2. Let k be a field and let $R = k[x, y]$ be the polynomial ring in two indeterminants x and y. Let $I = (x, y) \subset k[x, y]$ be the maximal ideal generated by x and y. Prove that I is a non-invertible fractional ideal and show $I \neq (I^{-1})^{-1}$.

3. Let k be any field. Let $R = k[x, y, z]/(z^2 - xy)$. Let $F = R_{(0)}$ be the field of fractions of the ring R.
(a) Convince yourself that R is a Noetherian domain and that both R_x and R_y are integrally closed inside F.
(b) Prove that there is an equality $R = R_x \cap R_y$ inside the fraction field of R. As R is then the intersection of integrally closed subrings of F, it follows that R is itself integrally closed.
(Hint: If $\omega \in R_x \cap R_y$ then we can write $\omega = f/x^m$ and $\omega = g/y^n$ for some elements $f, g \in R$ and some $n, m \geq 0$. Note that R is a free $k[x, y]$-module with basis $\{1, z\}$. Writing $f = f_0 + f_1 z$ and $g = g_0 + g_1 z$ for polynomials $f_0, f_1, g_0, g_1 \in k[x, y]$ it then follows that $y^n f_0 = x^m g_0$ and $y^n f_1 = x^m g_1$.

Conclude that $\omega \in R$.)

4. Let k be a field so that -1 is not the square of any element from k. Let E/k be the field extension which adjoins to k a square root of -1.
   (a) Show that $E[x,y]/(x^2+y^2-1)$ is isomorphic with $E[X,Y]/(XY-1)$ by associating X to $x+iy$ and Y to $x-iy$ where $i \in E$ is any element such that $i^2=-1$. Show then that $E[X,Y]/(XY-1)$ is isomorphic $E[t,t^{-1}]$ so that $E[x,y]/(x^2+y^2-1)$ is a Dedekind domain (in fact, a PID).
   (b) Let R be an integral domain with fraction field F. Suppose that a finite group G acts on F and let R^G denote the set of elements $x \in F$ so that $gx = x$ for all $g \in G$. Prove that if R is integrally closed in F, then R^G is integrally closed in its field of fractions as well.
   (c) Let G be the Galois group of E/k. Show that G has an action on the field of fractions $R = E[x,y]/(x^2+y^2-1)$ with $R^G = k[x,y]/(x^2+y^2-1)$ the subring of fixed elements. Deduce from this that R^G is a Dedekind domain.

5.* (Compare with the sources [Har77, §2, Ex. 6.4] and [Mat89, Example 4, p. 65]). Let R be any UFD with $1/2 \in R$ and let $f \in R$ be any squarefree element. Prove that $R[y]/(y^2-f)$ is an integrally closed domain.

6. In this exercise we'll show that the Picard group of the ring $\mathbb{Z}[\sqrt{-5}]$ is nontrivial by proving that $\mathrm{Pic}(\mathbb{Z}[\sqrt{-5}])$ has an element of order 2.
   (a) Check that $R = \mathbb{Z}[\sqrt{-5}]$ is integrally closed in its field of fractions $\mathbb{Q}(\sqrt{-5})$. Convince yourself that R is a Dedekind domain.
   (b) Define the *norm function* $N : R \to \mathbb{Z}$ by setting

$$N(a+b\sqrt{-5}) = a^2+5b^2 \quad \text{for } a,b \in \mathbb{Z}.$$

   Show that for any $x,y \in R$ we have $N(xy) = N(x)N(y)$. Show also that $N(u) = 1$ if and only if u is a unit in R.
   (c) Let $J = (2, 1+\sqrt{-5}) \subset R$ be the given ideal. Prove both that J is a prime ideal and that J is not principal. For the latter claim, note that if $J = (x)$ then x divides both 2 and $1+\sqrt{-5}$ so $N(x)$ divides both $N(2) = 4$ and $N(1+\sqrt{-5}) = 6$. Hence either $N(x) = 1$ (which would imply that x is a unit) or $N(x) = 2$ (which is impossible, since $a^2+5b^2 = 2$ has no solutions with $a,b \in \mathbb{Z}$).
   (d) Conclude that the class of the ideal J is a nontrivial element inside $\mathrm{Pic}(R)$. Check that $J^2 = (2) \subset R$ so that the class $[J] \in \mathrm{Pic}(R)$ has order 2.

   One way to see that there is an isomorphism $\mathrm{Pic}(\mathbb{Z}[\sqrt{-5}]) \cong \mathbb{Z}/2\mathbb{Z}$ is through use of Minkowski's bound for the ring of integers of an algebraic number field, [Mil14, Theorem 4.3 and Example 4.6]. For these rings, Minkowski's bound allows one to restrict the set of all possible representatives for the isomorphism classes of ideals in the Picard group to a specific (computable) finite set.

7. Let k be any field of characteristic not 2 and let $R = k[x, y, z]/(xy - z^2 + 1)$. Here we complete the computation from Example 1.2.16 and show both that $\mathrm{Pic}(R) \cong \mathbb{Z}$ and $\mathrm{Cl}(R) \cong \mathbb{Z}$.
   (a) Use Exercise 1.3.5 to show that R is integrally closed.
   (b) We're going to mimic the proof from Example 1.3.37 with some alterations. To start, observe that for $x \in R$ there are isomorphisms
   $$R/xR \cong k[y, z]/(z^2 - 1) \quad \text{and} \quad R_x \cong k[x, x^{-1}, z].$$
   It follows that the minimal primes of $(x) \subset R$ are the ideals $(x, z - 1)$ and $(x, z + 1)$ and that R_x is a UFD.
   (c) Show that there are isomorphisms of groups
   $$R_x^\times \cong \coprod_{i \in \mathbb{Z}} k^\times x^i \quad \text{and} \quad R^\times \cong k^\times.$$
   It follows that the canonical inclusion $R^\times \to R_x^\times$ has cokernel isomorphic with $\mathbb{Z}$. Hence the surjection $F^\times/R^\times \to F^\times/R_x^\times$ has kernel the subgroup generated by x which is isomorphic with $\mathbb{Z}$.
   (d) Show that $\mathrm{div}(x) = \mathfrak{p}_1 + \mathfrak{p}_2$ where $\mathfrak{p}_1 = (x, z-1)$ and $\mathfrak{p}_2 = (x, z+1)$. Make use of a commutative diagram like this one
   $$\begin{tikzcd} 1 \arrow[r] & F^\times/R^\times \arrow[r, "\mathrm{div}"] \arrow[d, equal] & \mathrm{WDiv}(R) \arrow[r] \arrow[d] & \mathrm{Cl}(R) \arrow[r] \arrow[d] & 0 \\ 1 \arrow[r] & F^\times/R_x^\times \arrow[r, "\mathrm{div}"] & \mathrm{WDiv}(R_y) \arrow[r] & \mathrm{Cl}(R_y) \arrow[r] & 0. \end{tikzcd}$$
   noting that the kernel of $\mathrm{WDiv}(R) \to \mathrm{WDiv}(R_x)$ is the subgroup $\mathbb{Z}\mathfrak{p}_1 \oplus \mathbb{Z}\mathfrak{p}_2$. Conclude that $\mathrm{Cl}(R) \cong \mathbb{Z}$.
   (e) Prove that $\mathfrak{p}_1$ (resp. $\mathfrak{p}_2$) is an invertible fractional ideal and prove that the map $\mathrm{div} : I_{fr}(R) \to \mathrm{WDiv}(R)$ satisfies $\mathrm{div}(\mathfrak{p}_1) = \mathfrak{p}_1$ (resp. $\mathrm{div}(\mathfrak{p}_2) = \mathfrak{p}_2$). Conclude that $\mathrm{Pic}(R) \cong \mathbb{Z}$. Note this proves also that R is locally factorial.

8. Unlike the Picard group which admits functorial maps for arbitrary morphisms of rings, see Exercise 1.2.7, the divisor class group is functorial only on certain restricted classes of morphisms of rings. Here we show that one can construct a group homomorphism between the divisor class groups of two rings given either a flat or integral extension of rings.
   (a)* Prove the going down theorem for flat ring extensions. More specifically, let R be any ring and let S be a ring with $R \subset S$ realizing S as a flat

R-module. Let $\mathfrak{p}_1 \subset \mathfrak{p}_2$ be two prime ideals of R and let $\mathfrak{q}_2$ be a prime ideal of S so that $\mathfrak{q}_2 \cap R = \mathfrak{p}_2$. Show that there exists a prime ideal $\mathfrak{q}_1 \subset \mathfrak{q}_2 \subset S$ so that $\mathfrak{q}_1 \cap R = \mathfrak{p}_1$. (Compare with [AM69, Ch. 3, Exercise 18].)

(b) Suppose that R and S are both Noetherian integrally closed domains with $R \subset S$ and assume that either S is a flat R-module or that $R \subset S$ is an integral extension of rings. Define a map

$$\overline{\mathrm{res}}_R^S : \mathrm{WDiv}(R) \to \mathrm{WDiv}(S) \qquad \mathfrak{p} \mapsto \sum_{\substack{\mathfrak{q} \subset S \text{ prime} \\ \mathfrak{q} \cap R = \mathfrak{p},\ \mathrm{ht}(\mathfrak{q})=1}} e(\mathfrak{q}/\mathfrak{p})\mathfrak{q}$$

where $e(\mathfrak{q}/\mathfrak{p})$ is the *ramification index* of $\mathfrak{p}$ in $\mathfrak{q}$. Note that, by definition, if π is a uniformizer for the local ring $S_\mathfrak{q}$ and if t is a uniformizer for the local ring $R_\mathfrak{p} \subset S_\mathfrak{q}$ then $e(\mathfrak{q}/\mathfrak{p}) = v_\pi(t)$. The definition of the map $\overline{\mathrm{res}}_R^S$ then makes sense because, for any prime ideal $\mathfrak{p} \subset R$ with $\mathrm{ht}(\mathfrak{p}) = 1$, there are only finitely many primes $\mathfrak{q} \subset S$ with both $\mathrm{ht}(\mathfrak{q}) = 1$ and $\mathfrak{q} \cap R = \mathfrak{p}$ (if $x \in \mathfrak{p}$ is any nonzero element then there are only finitely many minimal primes of S/xS since the latter is Noetherian).

Let F be the fraction field of R and let E be the fraction field of S. Write $\mathrm{div}_R : F^\times \to \mathrm{WDiv}(R)$ and $\mathrm{div}_S : E^\times \to \mathrm{WDiv}(S)$ for the two divisor maps. Use either part (a) or [AM69, Theorem 5.16] to show that $\overline{\mathrm{res}}_R^S \circ \mathrm{div}_R(f) = \mathrm{div}_S(f)$ for all $f \in F^\times$ so that there is a well-defined homomorphism

$$\mathrm{res}_R^S : \mathrm{Cl}(R) \to \mathrm{Cl}(S) \qquad [\mathfrak{p}] \mapsto \sum_{\substack{\mathfrak{q} \subset S \text{ prime} \\ \mathfrak{q} \cap R = \mathfrak{p},\ \mathrm{ht}(\mathfrak{q})=1}} e(\mathfrak{q}/\mathfrak{p})[\mathfrak{q}]$$

induced from the map $\overline{\mathrm{res}}_R^S$.

Prove that if R, S, T are three integrally closed Noetherian domains with $R \subset S \subset T$ and if either T is flat over S and S is flat over R, or if T is integral over S and S is integral over R, then T is flat over R, or T is integral over R, and $\mathrm{res}_R^T = \mathrm{res}_S^T \circ \mathrm{res}_R^S$.

9. Prove the following variant of Nagata's theorem: if R is an integrally closed Noetherian domain and if $S \subset R$ is a multiplicatively closed subset, then the kernel of the homomorphism from Exercise 1.3.8

$$\mathrm{res}_R^{S^{-1}R} : \mathrm{Cl}(R) \to \mathrm{Cl}(S^{-1}R)$$

is generated by classes of those prime ideals $\mathfrak{p} \subset R$ such that $\mathfrak{p} \cap S \neq \emptyset$. This observation has been tacitly used in Example 1.3.37 and Exercise 1.3.7.

10. If R and S are two integrally closed Noetherian domains, if $R \subset S$, and if S is flat as an R-module then the morphism $c_1 : \mathrm{Pic}(-) \to \mathrm{Cl}(-)$ is a natural transformation with respect to the maps res_R^S of Exercises 1.2.7 and 1.3.8.
   (a) Let $F = R_{(0)}$ be the fraction field of R, and suppose that $I \subset F$ is an invertible fractional ideal for R. Then there are elements $f_1, ..., f_n \in F$ so that $I^{-1} = f_1R + \cdots + f_nR$. Show that $I = f_1R \cap \cdots \cap f_nR$ and, if $E = S_{(0)}$ is the fraction field of S, show that the S-module $I \otimes_R S$ is isomorphic with the invertible fractional ideal $SI \subset E$ equal to $SI = f_1S \cap \cdots \cap f_nS$.
   (b) Let $\mathfrak{p} \subset R$ be a prime ideal with $\mathrm{ht}(\mathfrak{p}) = 1$. Let $f_1, ..., f_n \in F$ be elements so that $I = f_1R \cap \cdots \cap f_nR$ is an invertible fractional ideal. Prove that

$$\mathrm{ord}_{\mathfrak{p}}(I) = \max\{\mathrm{ord}_{\mathfrak{p}}(f_1), ..., \mathrm{ord}_{\mathfrak{p}}(f_n)\}.$$

   Use this to show that the map map

$$\overline{\mathrm{res}}_R^S : I_{fr}(R) \to I_{fr}(S) \qquad I \mapsto SI$$

   is a group homomorphism which fits into a commutative diagram

$$\begin{tikzcd} I_{fr}(R) \arrow[r, "\overline{\mathrm{res}}_R^S"] \arrow[d, "\mathrm{div}_R"] & I_{fr}(S) \arrow[d, "\mathrm{div}_S"] \\ \mathrm{WDiv}(R) \arrow[r, "\overline{\mathrm{res}}_R^S"] & \mathrm{WDiv}(S) \end{tikzcd}$$

   with the morphism $\overline{\mathrm{res}}_R^S : \mathrm{WDiv}(R) \to \mathrm{WDiv}(S)$ of Exercise 1.3.8.
   (c) Conclude that the following diagram commutes.

$$\begin{tikzcd} \mathrm{Pic}(R) \arrow[r, "\mathrm{res}_R^S"] \arrow[d, "c_1"] & \mathrm{Pic}(S) \arrow[d, "c_1"] \\ \mathrm{Cl}(R) \arrow[r, "\mathrm{res}_R^S"] & \mathrm{Cl}(S) \end{tikzcd}$$

   By Exercise 1.2.7, this means also that there is a commutative diagram like the following one:

$$\begin{tikzcd} K(R) \arrow[r, "\mathrm{res}_R^S"] \arrow[d, "c_1"] & K(S) \arrow[d, "c_1"] \\ \mathrm{Cl}(R) \arrow[r, "\mathrm{res}_R^S"] & \mathrm{Cl}(S). \end{tikzcd}$$

   Hence the first Chern class (Remark 1.3.30) is functorial with respect to flat ring extensions.

11. In this exercise we prove a generalization of the fact that a ring R is a UFD if and only if the ring $R[x]$ is a UFD. In particular, we show that: if R is an integrally closed Noetherian domain, then $R[x]$ is an integrally closed Noetherian domain and there is an isomorphism $\mathrm{Cl}(R) \cong \mathrm{Cl}(R[x])$.
   (a) Let R be an integrally closed Noetherian domain. Prove that $R[x]$, the ring of polynomials in one variable x with coefficients in R, is also an integrally closed Noetherian domain.
   (b) Let $\mathfrak{p} \subset R$ be a prime ideal with $\mathrm{ht}(\mathfrak{p}) = 1$. Assume that $\mathfrak{q} \subset R[x]$ is a prime ideal with $\mathfrak{q} \cap R = \mathfrak{p}$. Show that one of the following must be true:
      i. $\mathfrak{q} = \mathfrak{p}R[x]$ and $\mathrm{ht}(\mathfrak{q}) = 1$,
      ii. $\mathfrak{q} \supsetneq \mathfrak{p}R[x]$ and $\mathrm{ht}(\mathfrak{q}) > 1$.

   (Hint: in the latter case, show that there exists a $g \in \mathfrak{q}$ so that the image $\bar{\mathfrak{q}}$ of $\mathfrak{q}$ in $R[x]/\mathfrak{p}R[x] \cong (R/\mathfrak{p})[x]$ has the form

   $$\bar{\mathfrak{q}} = \{f \in (R/\mathfrak{p})[x] : af \in (g) \text{ for some } 0 \neq a \in R/\mathfrak{p}\}.$$

   One can take for g a polynomial of minimal degree in $\mathfrak{q}$.)
   (c) Let R be an integrally closed Noetherian domain with fraction field F. Let $S = R \setminus \{0\}$. Then $S^{-1}R[x] \cong F[x]$ is a PID and hence $\mathrm{Cl}(S^{-1}R[x]) = 0$. By Exercise 1.3.9, the restriction map

   $$\mathrm{res}_{R[x]}^{F[x]} : \mathrm{Cl}(R[x]) \to \mathrm{Cl}(F[x])$$

   has kernel generated by the prime ideals $\mathfrak{q}$ of $R[x]$ with $\mathfrak{q} \cap R \neq 0$. Use (b) from above to show that the map

   $$\mathrm{res}_{R}^{R[x]} : \mathrm{Cl}(R) \to \mathrm{Cl}(R[x])$$

   is surjective.
   (d) Suppose that $\mathfrak{p} \subset R$ is a prime ideal with $\mathrm{ht}(\mathfrak{p}) = 1$. Suppose that $\mathfrak{p}R[x]$ is a principal Weil divisor fR for some f in the fraction field F of R. Show that $\mathfrak{p}R[x] = (f)$ is then the principal ideal (f). Since $\mathfrak{p} \otimes_R R[x] \cong \mathfrak{p}R[x]$ it follows that

   $$\mathfrak{p} \cong \mathfrak{p} \otimes_R R[x] \otimes_{R[x]} R$$

   is principal, where in the last tensor R is treated as an $R[x]$-module via the isomorphism $R \cong R[x]/(x)$. Conclude that the map

   $$\mathrm{res}_{R}^{R[x]} : \mathrm{Cl}(R) \to \mathrm{Cl}(R[x])$$

   is injective as well.

12. (*Divisor class groups for more general rings*). Let k be a field and let R be an integral domain and finitely generated k-algebra with fraction field $F = R_{(0)}$. In this exercise we show how to define a group morphism $\operatorname{div} : F^\times \to \mathrm{WDiv}(R)$ which agrees with the map defined in Lemma 1.3.23 when R is integrally closed.

Let $R^\nu \subset F$ be the integral closure of R in F. The integral closure R^ν is a finitely generated R-module. In the case that $\mathrm{char}(F) = 0$, this can be proved as follows. By Noether's Normalization lemma, there is a k-subalgebra $A = k[y_1, ..., y_d] \subset R$ so that R is finitely generated as an A-module. Hence R^ν is the integral closure of A in the finite separable extension $F/k(y_1, ..., y_d)$ which is finitely generated by [AM69, Proposition 5.17]. The proof in the general case is similar, but must take into account the possibility of inseparable extensions, see [Ser00, §4, Proposition 16] or [Eis95, Corollary 13.3].

(a) Let $\mathfrak{p} \subset R$ be a prime ideal and write $R^\nu_\mathfrak{p}$ for the integral closure of $R_\mathfrak{p}$. Prove that there are finitely many prime ideals $\mathfrak{q} \subset R^\nu$ lying over $\mathfrak{p}$. Moreover, if $\mathfrak{q}$ is such an ideal then $\mathfrak{q}R^\nu_\mathfrak{p}$ is maximal in $R^\nu_\mathfrak{p}$ and the field extension degree $[R^\nu_\mathfrak{p}/\mathfrak{q}R^\nu_\mathfrak{p} : R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}]$ is finite.

(b) Define $\operatorname{div} : F^\times \to \mathrm{WDiv}(R)$ by

$$\operatorname{div}(f) = \sum_{\substack{\mathfrak{p}\subset R \text{ prime} \\ \mathrm{ht}(\mathfrak{p})=1}} \left(\sum_{\substack{\mathfrak{q}\subset R^\nu \text{ prime} \\ \mathfrak{q}\cap R=\mathfrak{p}}} \mathrm{ord}_\mathfrak{q}(f)[R^\nu_\mathfrak{p}/\mathfrak{q}R^\nu_\mathfrak{p} : R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}] \right) \cdot \mathfrak{p}.$$

Prove that div is a well-defined group homomorphism.

(c) Define $\mathrm{Cl}(R) = \mathrm{WDiv}(R)/\operatorname{div}(F^\times)$. Prove that if $R = k[x, y]/(y^2 - x^3)$ as in Exercise 1.2.10, then $\mathrm{Cl}(R) = 0$. If one defines $\operatorname{div} : I_{fr}(R) \to \mathrm{WDiv}(R)$ using a formula similar to the above, then this gives an example of a ring R so that $\mathrm{Pic}(R) \to \mathrm{Cl}(R)$ is not injective.

It's possible to define a "divisor map" $\operatorname{div} : F^\times \to \mathrm{WDiv}(R)$ for an arbitrary Noetherian ring R [Eis95, Theorem 11.10]. If R is either integrally closed or if R is a domain and a finitely generated k-algebra, then this more general definition agrees with either the definition from Lemma 1.3.23 or the definition from this exercise; in the latter case see [Ful98, Example 1.2.3].

1.4 G-theory

The G-theory $G(R)$ of a commutative ring R is an object that's more suitable for geometric questions. It's defined in a similar way to $K(R)$.

Definition 1.4.1. Let R be a commutative ring. Let $M_{fg}(R)$ be the free abelian group on isomorphism classes of finitely generated R-modules, i.e. let

$$M_{fg}(R) := \bigoplus_{M} \mathbb{Z} \cdot M$$

where the index M varies over the choice of a representative for each isomorphism class of finitely generated R-module. Let $M_{ex}(R) \subset M_{fg}(R)$ be the subgroup generated by elements $M - L - N$ for each short exact sequence

$$0 \to L \to M \to N \to 0$$

of finitely generated R-modules L, M, and N. We define the *G-theory of the ring R* as the quotient group $G(R) = M_{fg}(R)/M_{ex}(R)$.

Remark 1.4.2. Let $M_{lex}(R) \subset M_{fg}(R)$ be the subgroup generated by elements $\sum_{i\geq 1}(-1)^i N_i$ for each long exact sequence

$$0 \to N_r \to \cdots \to N_1 \to 0$$

of finitely generated R-modules. Set $N_k = 0$ for all $k \leq 0$ and for all $k \geq r+1$. Set $K_i = \ker(N_i \to N_{i-1})$. Then K_i is finitely generated as $K_i = \operatorname{Im}(N_{i+1} \to N_i)$ and there are short exact sequences

$$0 \to K_i \to N_i \to K_{i-1} \to 0$$

for all $i \geq 2$. The inclusion $M_{ex}(R) \subset M_{lex}(R)$ is then an isomorphism since

$$\sum_{i\geq 1}(-1)^i N_i = \sum_{i\geq 2}(-1)^i \left(N_i - K_i - K_{i-1}\right).$$

It follows that $G(R) = M_{fg}(R)/M_{ex}(R) = M_{fg}(R)/M_{lex}(R)$.

For any prime ideal $\mathfrak{p} \subset R$, the quotient $R/\mathfrak{p}$ is a finitely generated R-module and hence defines a class $[R/\mathfrak{p}]$ in $G(R)$. As the quotients $R/\mathfrak{p}$ are in one-to-one correspondence with closed subschemes of $\operatorname{Spec}(R)$, we can try to recover some information about the geometry of $\operatorname{Spec}(R)$ from these classes in $G(R)$. To do this formally, we introduce the following definition.

Definition 1.4.3. Let $X = \operatorname{Spec}(R)$ be the affine scheme associated to a ring R. We write $X^{(n)} \subset \operatorname{Spec}(R)$ for the set of prime ideals $\mathfrak{p} \subset R$ having $\operatorname{ht}(\mathfrak{p}) = n$. We then define $Z^n(R)$ as the free abelian group on symbols $R/\mathfrak{p}$ indexed by the elements of $X^{(n)}$, i.e.

$$Z^n(R) = \bigoplus_{\mathfrak{p}\in X^{(n)}} \mathbb{Z} \cdot R/\mathfrak{p}.$$

Elements of $Z^n(R)$ are called *height n-cycles* and the group $Z^n(R)$ is the *group of height n-cycles on R*. To denote the group of all cycles on R we'll write

$$Z(R) := \bigoplus_{n\in\mathbb{Z}_{\geq 0}} Z^n(R)$$

with the superscript n removed.

Remark 1.4.4. Theorem 1.3.19 (Krull's Principal Ideal Theorem) can be used to analyze the heights of ideals with a higher number of generators. Namely, if R is a Noetherian ring and if $I \subset R$ is an ideal which can be generated by n elements, then any minimal prime ideal $\mathfrak{p}$ over I has $\mathrm{ht}(\mathfrak{p}) \leq n$ (for proof one can see the sources [Eis95, Theorem 10.2] or [AM69, Corollary 11.16] or [Liu02, Ch. 2, Corollary 5.14]). This has the useful consequence that if R is a Noetherian ring, then every prime ideal $\mathfrak{p} \subset R$ has finite height.

Hence, for any Noetherian ring R, the group $Z(R)$ is canonically the free abelian group indexed by elements of $\mathrm{Spec}(R)$. Regardless of whether or not the ring R is Noetherian, there is also a canonical isomorphism $\mathrm{WDiv}(R) \cong Z^1(R)$ making the connection between prime ideals $\mathfrak{p}$ and quotients $R/\mathfrak{p}$.

Now there is a well-defined homomorphism

$$cl : Z(R) \to G(R) \quad \text{defined by} \quad cl(R/\mathfrak{p}) = [R/\mathfrak{p}] \tag{1.4.5}$$

and extended linearly. We want to see how well this map describes the group $G(R)$ and the first step in this regard is:

Proposition 1.4.6. *Let $R \neq 0$ be a Noetherian ring. Then the homomorphism*

$$cl : Z(R) \to G(R)$$

of (1.4.5) is a surjection.

Before proving the proposition, we prove the following lemma which will be used both in the proof and a few times throughout the remainder of this section.

Lemma 1.4.7. *Let L, M, N be three R-modules fitting into an exact sequence*

$$0 \to L \xrightarrow{i} M \xrightarrow{\pi} N \to 0.$$

Suppose that $N' \subset N$ is any R-submodule with inclusion $j : N' \subset N$. Then there *is an R-submodule $M' \subset M$ and a commutative diagram*

(1.4.8)
$$\begin{tikzcd} & & 0 & 0 & \\ & & M/M' \arrow[u] \arrow[r, equal] & N/N' \arrow[u] & \\ 0 \arrow[r] & L \arrow[r, "i"] & M \arrow[u] \arrow[r, "\pi"] & N \arrow[u] \arrow[r] & 0 \\ 0 \arrow[r] & L \arrow[u, equal] \arrow[r] & M' \arrow[u] \arrow[r] & N' \arrow[u, "j"] \arrow[r] & 0 \\ & & 0 \arrow[u] & 0 \arrow[u] & \end{tikzcd}$$

with both horizontal rows, and vertical columns, exact sequences.

Proof. We set $M' = M \times_N N'$ to be the fiber product of R-modules, i.e. M' is the subset of the product $M \times N'$ consisting of all pairs (a, b) with $\pi(a) = j(b)$. Then M' is naturally an R-submodule of the product $M \times N'$ and composing with the two projections to M and N' form the respective vertical and horizontal arrows from M' in (1.4.8).

We leave it as an exercise for the reader to check that the diagram commutes and that both the rows and columns are exact (see Exercise 1.4.2); we also point out that M' is naturally identified with the preimage $\pi^{-1}(N') \subset M$. $\square$

Proof of Proposition 1.4.6. Let $M \neq 0$ be a finitely generated R-module. Consider

$$\mathrm{Ann}_R(x) = \{f \in R : fx = 0\},$$

i.e. the ideal of R which is the annihilator for the element x of M. The collection of ideals $\{\mathrm{Ann}_R(x)\}_{x \in M \setminus \{0\}}$, where $x \in M$ ranges over all nonzero elements, has a maximal element since R is Noetherian. Let $\mathfrak{p}_1 = \mathrm{Ann}_R(y_1)$ be any such maximal element for some nonzero $y_1 \in M$. Then $\mathfrak{p}_1 \subsetneq R$ is a prime ideal: if $f, g \in R$ are such that $(fg)y_1 = 0$, then either $gy_1 = 0$ or $gy_1 \neq 0$; in the latter case there's a containment $\mathrm{Ann}_R(gy_1) \supset \mathrm{Ann}_R(y_1)$, which must actually be an equality by our choice of y_1, so that $f(gy_1) = 0$ implies $f \in \mathfrak{p}_1$ as desired.

Setting $M_1 = R/\mathfrak{p}_1$, we find a short exact sequence

$$0 \to M_1 \xrightarrow{1 \mapsto y_1} M \to N_1 \to 0$$

with N_1 the appropriate cokernel. If $N_1 \neq 0$, then as before we can find a prime ideal of the form $\mathfrak{p}_2 = \mathrm{Ann}_R(y_2)$ but now for some y_2 in N_1. Let $M_2 \subset M$ be the preimage of $R/\mathfrak{p}_2 \subset N_1$. We have that $M_2/M_1 \cong R/\mathfrak{p}_2$ by Lemma 1.4.7. Repeating this process we get a sequence of submodules of M

$$0 = M_0 \subset M_1 \subset \cdots \subset M_i \subset \cdots \subset M$$

with associated quotients $M_{i+1}/M_i \cong R/\mathfrak{p}_{i+1}$ and with $N_i := N_{i-1}/(M_{i+1}/M_i)$. Since M is finitely generated and R is Noetherian, the chain of M_i's must stabilize after some $n \gg 0$ steps; at this point we must also have $N_n = 0$ because if there was a nonzero $z \in N_n$ the ideal $\mathrm{Ann}_R(z) \subsetneq R$ would be proper and $M_{n+1}/M_n \neq 0$. Altogether this gives an equality

$$\begin{aligned} [M] &= [N_1] + [R/\mathfrak{p}_1] \\ &\vdots \\ &= [N_k] + [R/\mathfrak{p}_k] + \cdots + [R/\mathfrak{p}_1] \\ &\vdots \\ &= [R/\mathfrak{p}_n] + \cdots + [R/\mathfrak{p}_1] \end{aligned}$$

inside of $G(R)$. As we've shown an arbitrary generator of $G(R)$ can be written as a sum of elements in the image $cl(Z(R))$, this completes the proof. □

We'll come back to an observation made in this proof momentarily, but first we see some examples.

Remark 1.4.9. Let R be an integral domain. Set $F = R_{(0)}$ to be the field of fractions of R. The assignment

$$\mathrm{rk} : M_{fg}(R) \to \mathbb{Z} \qquad M \mapsto \dim_F(M_{(0)}),$$

sending an R-module M to the dimension of the localization $M_{(0)}$ as an F-vector space, is zero on $M_{ex}(R)$. It therefore defines a homomorphism

$$\mathrm{rk} : G(R) \to \mathbb{Z}$$

which we call *the rank homomorphism for R.*

Example 1.4.10. Suppose that R is a PID. The same argument that was used in Example 1.2.5 but, using the rank homomorphism for $G(R)$ instead of $K(R)$, shows that $G(R) = \mathbb{Z}$ with the class $[R]$ an additive generator.

Example 1.4.11. Let $R = \mathbb{Z}[x]$ be the integral polynomial ring in one variable. The Krull dimension of R is 2 and we can describe the sets $\mathrm{Spec}(R)^{(n)}$ explicitly for each of $n = 0, 1, 2$. They are:

$$\begin{aligned}
&\mathrm{Spec}(R)^{(0)} = \{(0)\} \\
&\mathrm{Spec}(R)^{(1)} = \{(p), (f(x))\}_{p,f} && \text{with } p \text{ prime or } f(x) \text{ irreducible} \\
&\mathrm{Spec}(R)^{(2)} = \{(p, f(x))\}_{p,f} && \text{with } f(x) \text{ irreducible modulo } p.
\end{aligned}$$

By Proposition 1.4.6, it follows that $G(R)$ is generated by the classes of

$$\mathbb{Z}[x], \quad \mathbb{F}_p[x], \quad \mathbb{Z}[x]/f(x), \quad \text{and} \quad \mathbb{F}_q$$

where $\mathbb{F}_q$ is the finite field of q elements for a power $q = p^n$ of a prime p and with $n \geq 1$. But, there are short exact sequences

$$0 \to \mathbb{Z}[x] \xrightarrow{1 \mapsto p} \mathbb{Z}[x] \to \mathbb{F}_p[x] \to 0$$
$$0 \to \mathbb{Z}[x] \xrightarrow{1 \mapsto f(x)} \mathbb{Z}[x] \to \mathbb{Z}[x]/(f(x)) \to 0$$
$$0 \to \mathbb{F}_p[x] \xrightarrow{1 \mapsto f(x)} \mathbb{F}_p[x] \to \mathbb{F}_q \to 0$$

which show that

$$[\mathbb{F}_p[x]] = 0, \quad [\mathbb{Z}[x]/(f(x))] = 0, \quad \text{and} \quad [\mathbb{F}_q] = 0$$

inside of $G(R)$. Since $\mathrm{rk}([\mathbb{Z}[x]]) \neq 0$, it follows that $G(R) = \mathbb{Z}$.

Returning to part of the proof of Proposition 1.4.6, we observed there that for any given Noetherian ring $R \neq 0$, each finitely generated R-module M admits a finite ascending filtration $M_\bullet$ by R-submodules,

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M,$$

with the property that $M_{i+1}/M_i \cong R/\mathfrak{p}_{i+1}$ for some prime ideals $\mathfrak{p}_1$, ..., $\mathfrak{p}_n$ of R. This type of filtration is useful enough to have a name:

Definition 1.4.12. Let M be an R-module for a ring $R \neq 0$. A finite ascending filtration $M_\bullet$ of M,

$$M_\bullet \equiv (0 = M_0 \subset M_1 \subset \cdots \subset M_n = M)$$

is called a *prime filtration of M* if for each $i \geq 0$ there is a prime ideal $\mathfrak{p}_{i+1} \subset R$ and an isomorphism $M_{i+1}/M_i \cong R/\mathfrak{p}_{i+1}$ of R-modules.

Given a prime filtration $M_\bullet$ of M as above, we write

$$\mathrm{cyc}(M_\bullet) = \sum_{i=1}^{n} [R/\mathfrak{p}_i] \in Z(R)$$

and call this sum the *cycle associated to the prime filtration $M_\bullet$*.

Let $\mathfrak{p} \subset R$ be a prime ideal and pick an element $f \in R \setminus \mathfrak{p}$ in the complement. If we set $I = \mathfrak{p} + (f)$ then there is a short exact sequence

$$0 \to R/\mathfrak{p} \xrightarrow{x \mapsto xf} R/\mathfrak{p} \to R/I \to 0$$

showing that $[R/I] = 0$ inside $G(R)$. In particular, for any prime filtration $R/I_\bullet$ of R/I, the cycle $\mathrm{cyc}(R/I_\bullet)$ has trivial class in $G(R)$.

We can generate all relations on $Z(R)$ that are needed to describe the quotient $G(R)$ by considering cycles of this type only. Specifically, let $\partial Z(R) \subset Z(R)$ be the subgroup generated by all cycles $\mathrm{cyc}(F_\bullet)$ coming from a prime filtration $F_\bullet$ of a quotient $R/(\mathfrak{p} + (f))$ and varying over all prime ideals $\mathfrak{p} \subset R$ and all $f \in R \setminus \mathfrak{p}$. Then there is the following:

Proposition 1.4.13. *Let $R \neq 0$ be a Noetherian ring. Then the surjection*

$$cl : Z(R) \to G(R)$$

of (1.4.5) *has kernel $\partial Z(R)$.*

We've seen that the map $cl : Z(R) \to G(R)$ descends to a map on the quotient $Z(R)/\partial Z(R) \to G(R)$. The proof works by constructing an inverse to this map. The most obvious choice for an inverse would be to send the class of a module $[M] \in G(R)$ to the equivalence class $[\mathrm{cyc}(M_\bullet)]$ in $Z(R)/\partial Z(R)$ for any choice of prime filtration $M_\bullet$ of M. Since there's no canonical choice for a prime filtration of an arbitrary module, it takes some work to check that this is well-defined.

Lemma 1.4.14. *Let $R \neq 0$ be a Noetherian ring and let M be a finitely generated R-module. Then, given any two prime filtrations $A_\bullet$ and $B_\bullet$ of M, there exist prime filtrations $A'_\bullet$ and $B'_\bullet$, refining $A_\bullet$ and $B_\bullet$ respectively, such that* $\mathrm{cyc}(A'_\bullet) = \mathrm{cyc}(B'_\bullet)$.

(Here a filtration $A'_\bullet$ of M

$$A'_\bullet \equiv (0 = A'_0 \subset A'_1 \subset \cdots \subset A'_m = M)$$

is said to be a refinement of a filtration

$$A_\bullet \equiv (0 = A_0 \subset A_1 \subset \cdots A_n = M)$$

if for all integers i with $0 \leq i \leq n$ there is an integer j so that $A_i = A'_j$).

Proof. Let's write $A_\bullet$ as

$$0 = A_0 \subset A_1 \subset \cdots \subset A_n = M$$

and $B_\bullet$ as

$$0 = B_0 \subset B_1 \subset \cdots \subset B_m = M$$

for some $n, m \geq 0$. We can then refine $A_\bullet$ to a filtration $A'_\bullet$ by inserting terms, for all $0 \leq i < n$, of the form

$$A_i = A_i + (A_{i+1} \cap B_0) \subset A_i + (A_{i+1} \cap B_1) \subset \cdots \subset A_i + (A_{i+1} \cap B_m) = A_{i+1}.$$

Similarly we can refine $B_\bullet$ to a filtration $B'_\bullet$ by inserting, for each $0 \leq j < m$, terms in the same way

$$B_j = B_j + (B_{j+1} \cap A_0) \subset B_j + (B_{j+1} \cap A_1) \subset \cdots \subset B_j + (B_{j+1} \cap A_m) = B_{j+1}.$$

Now the Butterfly lemma implies that, for fixed i, j in the given range, we have isomorphisms

$$\frac{A_i + (A_{i+1} \cap B_{j+1})}{A_i + (A_{i+1} \cap B_j)} \cong \frac{B_j + (B_{j+1} \cap A_{i+1})}{B_j + (B_{j+1} \cap A_i)}$$

of R-modules which means that the two filtrations $A'_\bullet$ and $B'_\bullet$ have the same associated quotients.

At the moment, neither of the new filtrations $A'_\bullet$ nor $B'_\bullet$ is necessarily a prime filtration of M. To fix this, let's assume that we've fixed indices i and j together with an isomorphism $A'_{i+1}/A'_i \cong B'_{j+1}/B'_j$. After possibly eliminating any repeating terms in the filtrations $A'_\bullet$ and $B'_\bullet$, we can assume that these quotients are nonzero. Hence we can fix an inclusion of some quotient $R/\mathfrak{p}$ for some prime $\mathfrak{p} \subset R$ into both A'_{i+1}/A'_i and B'_{j+1}/B'_j simultaneously, as in the proof of Proposition 1.4.6.

If we let A''_1 be the preimage of $R/\mathfrak{p}$ under the projection $A'_{i+1}/A'_i \to A'_{i+1}$, and let B''_1 be similarly the preimage in B'_{j+1}, then we have inclusions

$$A'_i \subset A''_1 \subset A'_{i+1} \quad \text{and} \quad B'_j \subset B''_1 \subset B'_{j+1}.$$

The quotients A''_1/A'_i and B''_1/B'_j are both isomorphic with $R/\mathfrak{p}$ by Lemma 1.4.7. Since $A'_{i+1}/A''_1 \cong B'_{j+1}/B''_1$ by Lemma 1.4.7 as well, we can continue this process until we have chains

$$A'_i \subset A''_1 \subset \cdots \subset A''_r \subset A'_{i+1} \quad \text{and} \quad B'_j \subset B''_1 \subset \cdots \subset B''_r \subset B'_{j+1}$$

with isomorphic associated quotients and so that all associated quotients have the form $R/\mathfrak{p}$ for possibly varying prime ideals $\mathfrak{p} \subset R$. Doing this at every step of the filtrations $A'_\bullet$ and $B'_\bullet$ allows us to construct new filtrations, say $A''_\bullet$ and $B''_\bullet$ with the desired properties. □

Lemma 1.4.15. *Let R be any ring and let M be an R-module. Suppose that $A_\bullet$ is any prime filtration of M and suppose $A'_\bullet$ is a prime filtration of M refining $A_\bullet$. Then*

$$[\mathrm{cyc}(A_\bullet)] = [\mathrm{cyc}(A'_\bullet)] \quad \textit{inside} \quad Z(R)/\partial Z(R).$$

Proof. Let's write $A_\bullet$ as

$$0 = A_0 \subset A_1 \subset \cdots \subset A_n = M$$

and let's fix an index $0 \leq i < n$ where the inclusion $A_i \subset A_{i+1}$ is refined. In other words, in the filtration $A'_\bullet$ there is an index j and an integer $k > 1$ so that

$$A_i = A'_j \subset A'_{j+1} \subset \cdots \subset A'_{j+k-1} \subset A'_{j+k} = A_{i+1}. \tag{1.4.16}$$

Our goal is to compare the associated quotient A_{i+1}/A_i of the filtration $A_\bullet$ and the quotients A'_{j+r+1}/A'_{j+r} for all $0 \leq r < k$ of the filtration $A'_\bullet$.

Taking the quotient of each term in the chain (1.4.16) by A'_j produces a new chain with the same associated quotients

$$0 = \frac{A'_j}{A'_j} \subset \frac{A'_{j+1}}{A'_j} \subset \cdots \subset \frac{A'_{j+k-1}}{A'_j} \subset \frac{A'_{j+k}}{A'_j} \cong \frac{A_{i+1}}{A_i}. \tag{1.4.17}$$

Since both $A_\bullet$ and $A'_\bullet$ were prime filtrations, we have that $A_{i+1}/A_i \cong R/\mathfrak{p}$ and $A'_{j+1}/A'_j \cong R/\mathfrak{q}$ for some prime ideals $\mathfrak{p}, \mathfrak{q} \subset R$. But, for any two ideals $I, J \subset R$, an inclusion $R/I \subset R/J$ of R-modules can happen if and only if $I = J$.

This means there is a commutative diagram with exact rows as below.

$$\begin{tikzcd}
0 \arrow[r] & A'_{j+1}/A'_j \arrow[r] \arrow[d, equal] & A'_{j+k}/A'_j \arrow[r] \arrow[d, equal] & A'_{j+k}/A'_{j+1} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & R/\mathfrak{p} \arrow[r, "1 \mapsto f"] & R/\mathfrak{p} \arrow[r] & R/(\mathfrak{p} + (f)) \arrow[r] & 0
\end{tikzcd}$$

If we quotient terms in the chain of 1.4.17 by A'_{j+1}/A'_j, and omit the first zero term, we end up with a chain like

$$0 = \frac{A'_{j+1}}{A'_{j+1}} \subset \frac{A'_{j+2}}{A'_{j+1}} \subset \cdots \subset \frac{A'_{j+k}}{A'_{j+1}} \cong R/(\mathfrak{p} + (f)). \tag{1.4.18}$$

Now the chain in 1.4.18 has all the same associated quotients as 1.4.17 except for the first one that was isomorphic with $R/\mathfrak{p}$. In particular, the chain in 1.4.18 is a prime filtration for $R/(\mathfrak{p} + (f))$.

Altogether this shows that the associated quotients of 1.4.16 are precisely one copy of $R/\mathfrak{p}$ together with the quotients associated with some prime filtration for some module of the form $R/(\mathfrak{p} + (f))$. Hence the difference $\mathrm{cyc}(A'_\bullet) - \mathrm{cyc}(A_\bullet)$ is made up exactly of sums of those cycles generating $\partial Z(R)$. $\square$

Proof of Proposition 1.4.13. Define a map $M_{fg}(R) \to Z(R)$ by sending $[M]$ to any cycle $\mathrm{cyc}(M_\bullet)$ for a fixed prime filtration $M_\bullet$ of M. Projecting to the quotient gives a map

$$M_{fg}(R) \to Z(R)/\partial Z(R)$$

which is independent of this choice. Indeed, if $A_\bullet$ was another prime filtration of M then by Lemma 1.4.14 there are refinements $A'_\bullet$ of $A_\bullet$ and $M'_\bullet$ of $M_\bullet$ so that $\mathrm{cyc}(A'_\bullet) = \mathrm{cyc}(M'_\bullet)$ in $Z(R)$. Applying Lemma 1.4.15 twice, we see that

$$[\mathrm{cyc}(A_\bullet)] = [\mathrm{cyc}(A'_\bullet)] = [\mathrm{cyc}(M'_\bullet)] = [\mathrm{cyc}(M_\bullet)] \in Z(R)/\partial Z(R)$$

showing independence of any choices.

Next, we observe the map $M_{fg}(R) \to Z(R)/\partial Z(R)$ induces map from $G(R)$ by checking that the image of $M_{ex}(R)$ is trivial in $Z(R)/\partial Z(R)$. So assume there is a short exact sequence

$$0 \to L \to M \to N \to 0$$

of finitely generated R-modules and fix prime filtrations $L_\bullet$ of L and $N_\bullet$ of N. We'll show that $L_\bullet$ and $N_\bullet$ induce a prime filtration $M_\bullet$ of M so that there is an equality $\mathrm{cyc}(M_\bullet) = \mathrm{cyc}(N_\bullet) + \mathrm{cyc}(L_\bullet)$ as cycles in $Z(R)$.

Write $L_\bullet$ as

$$0 = L_0 \subset \cdots \subset L_n = L$$

and $N_\bullet$ as

$$0 = N_0 \subset \cdots \subset N_m = N.$$

Then applying Lemma 1.4.7 with $N' = N_{m-1}$ shows that the preimage of N_{m-1} in M is an R-submodule $M_{n+m-1} \subset M$ with isomorphisms $M/M_{n+m-1} \cong N_m/N_{m-1}$ and $M_{n+m-1}/L \cong N_{m-1}$. Applying 1.4.7 again, now to the short exact sequence

$$0 \to L \to M_{n+m-1} \to N_{m-1} \to 0$$

and the inclusion $N_{m-2} \subset N_{m-1}$, produces an R-submodule $M_{n+m-2} \subset M_{n+m-1}$ with $M_{n+m-1}/M_{n+m-2} \cong N_{m-1}/N_{m-2}$ and $M_{n+m-2}/L \cong N_{m-2}$. Continuing we get a chain of R-submodules making up part of a prime filtration $M_\bullet$ of M

$$L = M_n \subset \cdots \subset M_{n+m} = M$$

with quotients $M_{n+j+1}/M_{n+j} \cong N_{j+1}/N_j$ for all $0 \leq j < m$. We complete the construction of $M_\bullet$ by setting $M_i = L_i$ for all $0 \leq i < n$.

Finally, it follows that there is a well-defined morphism $G(R) \to Z(R)/\partial Z(R)$ sending $[M]$ to the class $[\mathrm{cyc}(M_\bullet)]$ for any prime filtration $M_\bullet$ of M. Since this map is clearly inverse to the morphism $Z(R)/\partial Z(R) \to G(R)$ sending $[R/\mathfrak{p}]$ to the class $[R/\mathfrak{p}]$, we are done. □

The next theorem shows that if R is an integrally closed Noetherian domain, then we can recover the divisor class group $\mathrm{Cl}(R)$ as a subquotient of $G(R)$ using the description of $G(R)$ as a quotient of the group of cycles $Z(R)$. This result can be viewed as the first true hint that the group $G(R)$ is intimately tied to geometry, however, it is only one piece of a much broader picture.

Theorem 1.4.19. *Let $R \neq 0$ be an integrally closed Noetherian domain. Then there are subgroups*
$$F^2_\tau G(R) \subset F^1_\tau G(R) \subset G(R)$$
and canonical isomorphisms
$$G(R)/F^1_\tau G(R) \cong \mathbb{Z}$$
$$F^1_\tau G(R)/F^2_\tau G(R) \cong \mathrm{Cl}(R).$$

Proof. We set $F^1_\tau G(R)$ to be the subgroup of $G(R)$ generated by all of the images $cl(Z^n(R))$ for every $n \geq 1$. Then we have a commutative diagram with exact rows and surjective vertical arrows.

$$\begin{tikzcd} 0 \arrow[r] & \bigoplus_{n\geq 1} Z^n(R) \arrow[r] \arrow[d, "cl"] & Z(R) \arrow[r] \arrow[d, "cl"] & \mathbb{Z}\cdot R \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & F^1_\tau G(R) \arrow[r] & G(R) \arrow[r, "\mathrm{rk}"] & \mathbb{Z} \arrow[r] & 0 \end{tikzcd} \tag{1.4.20}$$

Exactness of the bottom row is a consequence of the fact $R/\mathfrak{p} \otimes_R R_{(0)} \cong 0$ if $\mathfrak{p} \subset R$ is any nonzero prime ideal. This gives the first isomorphism of the theorem.

Similarly, we set $F^2_\tau G(R) \subset F^1_\tau G(R)$ to be the subgroup of $G(R)$ generated by the groups $cl(Z^n(R))$ for all $n \geq 2$. We again have a commutative diagram with exact rows and surjective vertical arrows

$$\begin{tikzcd} 0 \arrow[r] & \bigoplus_{n\geq 2} Z^n(R) \arrow[r] \arrow[d, "cl"] & \bigoplus_{n\geq 1} Z^n(R) \arrow[r] \arrow[d, "cl"] & Z^1(R) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & F^2_\tau G(R) \arrow[r] & F^1_\tau G(R) \arrow[r] & F^1_\tau G(R)/F^2_\tau G(R) \arrow[r] & 0 \end{tikzcd}$$

and, by Remark 1.4.4, the group $Z^1(R) \cong \mathrm{WDiv}(R)$. Note, because of the diagram (1.4.20) above, the kernel of the map
$$cl : \bigoplus_{n\geq 1} Z^n(R) \to F^1_\tau G(R)$$
is isomorphic to $\partial Z(R)$. We'll show that the relations $\partial Z(R)$ in $Z^1(R) \cong \mathrm{WDiv}(R)$ generate a subgroup isomorphic with $\mathrm{div}(F^\times)$ where $F = R_{(0)}$ is the fraction field of R giving the isomorphism with $\mathrm{Cl}(R)$.

The subgroup $\partial Z(R)$ is generated by cycles of the form $\mathrm{cyc}(F_\bullet)$ coming from a prime filtration $F_\bullet$ of a quotient $R/(\mathfrak{p} + (f))$ and varying over all prime filtrations, prime ideals $\mathfrak{p} \subset R$, and elements $f \in R \setminus \mathfrak{p}$. Given one such cycle, say
$$\mathrm{cyc}(F_\bullet) = \sum_{i=1}^{n} [R/\mathfrak{p}_i] \in \bigoplus_{n\geq 1} Z^n(R),$$

the image of $\mathrm{cyc}(F_\bullet)$ in $Z^1(R)$ is the sum of those summands $[R/\mathfrak{p}_i]$ with prime ideals $\mathfrak{p}_i \subset R$ having $\mathrm{ht}(\mathfrak{p}_i) = 1$. If $\mathrm{ht}(\mathfrak{p}) \geq 1$, then any prime ideal $\mathfrak{p}_i$ appearing in such a prime filtration $F_\bullet$ contains $\mathfrak{p}$ strictly and so has height at least 2. Thus the image of all such cycles vanishes in $Z^1(R)$.

This means that the kernel of the map

$$Z^1(R) \to F^1_\tau G(R)/F^2_\tau G(R)$$

is generated by the images of cycles $\mathrm{cyc}(F_\bullet)$ where $F_\bullet$ is a prime filtration of a quotient R/fR for some element $f \in R \setminus \{0\}$. So let $\mathfrak{p} \subset R$ be a prime ideal with $\mathrm{ht}(\mathfrak{p}) = 1$, let $f \in R \setminus \{0\}$ be given, and let $F_\bullet$ be a prime filtration of R/fR. Then, localizing the filtration $F_\bullet$ at $\mathfrak{p}$ produces a filtration $(F_\mathfrak{p})_\bullet$ of $R_\mathfrak{p}/fR_\mathfrak{p}$ whose associated quotients are either 0, $R_\mathfrak{p}$, or $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} \cong R/\mathfrak{p}$. Moreover, the number of occurrences of the quotient $R/\mathfrak{p}$ in the two filtrations $F_\bullet$ and $(F_\mathfrak{p})_\bullet$ are the same.

We claim that the number of times $R/\mathfrak{p}$ occurs as a quotient from the filtration $(F_\mathfrak{p})_\bullet$ is uniquely determined by f and equal to $\mathrm{ord}_\mathfrak{p}(f)$. If this were true then the class of $\mathrm{cyc}(F_\bullet)$ inside $Z^1(R)$ is exactly the sum

$$\sum_{\substack{\mathfrak{p} \subset R \\ \mathrm{ht}(\mathfrak{p})=1}} \mathrm{ord}_\mathfrak{p}(f) R/\mathfrak{p}.$$

As cycles of this type, varying over all $f \in R \setminus \{0\}$, generate the corresponding subgroup $\mathrm{div}(F^\times) \subset \mathrm{WDiv}(R)$ this would complete the proof.

The filtration $(F_\mathfrak{p})_\bullet$ is an ascending filtration of $R_\mathfrak{p}/fR_\mathfrak{p}$ which starts with 0, ends with $R_\mathfrak{p}/(\pi^r)$, and which has associated quotients either 0, $R_\mathfrak{p}$, or $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$. Since $R_\mathfrak{p}$ is a DVR, as R is integrally closed, we can pick a uniformizer π for $\mathfrak{p}R_\mathfrak{p}$ and write $f = u\pi^r$ for some unit $u \in R_\mathfrak{p}$ and for some $r \geq 0$. This allows us to write $R_\mathfrak{p}/fR_\mathfrak{p} \cong R_\mathfrak{p}/(\pi^r)$. As an $R_\mathfrak{p}$-module, any ascending filtration of $R_\mathfrak{p}/(\pi^r)$ which starts with 0 and ends with $R_\mathfrak{p}/(\pi^r)$ and which has associated quotients either 0, $R_\mathfrak{p}$, or $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} \cong R/\mathfrak{p}$ contains exactly r-terms with associated quotients $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} \cong R/\mathfrak{p}$. But $\mathrm{ord}_\mathfrak{p}(f) = v_\pi(f) = r$ as claimed. □

Corollary 1.4.21. *Let R be a Dedekind domain. Then $G(R) \cong \mathbb{Z} \oplus \mathrm{Cl}(R)$.*

Proof. In the proof of Theorem 1.4.19, we observed that the subgroup $F^2_\tau G(R)$ of $G(R)$ was generated by classes $[R/\mathfrak{p}]$ for prime ideals $\mathfrak{p} \subset R$ of with $\mathrm{ht}(\mathfrak{p}) > 1$. However, in a Dedekind domain there are no such prime ideals so $F^2_\tau G(R) = 0$. □

Exercises for Section 1.4

1. Prove that $G(R) = 0$ if $R = 0$ is the zero ring.

2. Complete the proof of Lemma 1.4.7. In the notation of the lemma, prove also that M' can be identified with the preimage $\pi^{-1}(N') \subset M$.

3. Let R be a ring and let $\mathcal{C}_\bullet$ be a bounded complex of R-modules (recall that for a complex $\mathcal{C}_\bullet$ given by a collection of R-modules C_i, specified for all integers $i \in \mathbb{Z}$, together with a collection of morphisms $d_i : C_i \to C_{i-1}$ satisfying $d_i \circ d_{i+1} = 0$, then $\mathcal{C}_\bullet$ is said to be *bounded* if there exists an integer $N \geq 0$ so that $C_i = 0$ for all integers i with $|i| \geq N$). Show that there is an equality inside $G(R)$

$$\sum_{i\in\mathbb{Z}}(-1)^i[C_i] = \sum_{i\in\mathbb{Z}}(-1)^i[\mathrm{H}_i(\mathcal{C}_\bullet)]$$

between the alternating sum of the terms of $\mathcal{C}_\bullet$ and the alternating sum of the homology of $\mathcal{C}_\bullet$.

4. Let $f : R \to S$ be a morphism of rings R and S which realizes S as an R-module.
   (a) Assume that S is finitely generated as an R-module. Show that there is a well-defined group homomorphism

   $$\mathrm{cor}^R_S : G(S) \to G(R)$$

   induced by the map $M_{fg}(S) \to M_{fg}(R)$ which sends an S-module M to M considered as an R-module via f.

   If $g : S \to T$ is another morphism of rings which realizes T as a finitely generated S-module, then show that T is a finitely generated R-module. Prove there is an equality $\mathrm{cor}^R_S \circ \mathrm{cor}^S_T = \mathrm{cor}^R_T$.

   (b) Assume instead that S is flat as an R-module. Show that there is a well-defined group homomorphism

   $$\mathrm{res}^S_R : G(R) \to G(S)$$

   induced by the map $M_{fg}(R) \to M_{fg}(S)$ which sends an R-module M to the S-module $M \otimes_R S$.

   If $g : S \to T$ is another morphism of rings which realizes T as a flat S-module, then show that T is a flat R-module. Prove there is an equality $\mathrm{res}^T_S \circ \mathrm{res}^S_R = \mathrm{res}^T_R$.

5. (*Projection formula*). Let R be a ring. Consider the map

$$\mu : K(R) \to \mathrm{End}(G(R)) \qquad [P] \mapsto ([M] \mapsto [M \otimes_R P])$$

from $K(R)$ to the endomorphism ring of the group $G(R)$. Prove that the map μ is a ring homomorphism. Hence the group $G(R)$ is naturally a $K(R)$-module.

Let S be another ring with $f : R \to S$ a ring homomorphism. Assume that S is a finitely generated R-module. Show that for any two elements $x \in K(R)$ and $y \in G(R)$ there is an equality

$$\operatorname{cor}_S^R\left(\operatorname{res}_R^S(x) \cdot y\right) = x \cdot \operatorname{cor}_S^R(y)$$

using the map $\operatorname{res}_R^S : K(R) \to K(S)$ of Exercise 1.2.3.

6. Let R be a commutative Noetherian ring and let $I \subset R$ be a nilpotent ideal, i.e. an ideal having the property that $I^n = 0$ for some $n \geq 1$.
   (a) Prove that the homomorphism

   $$\operatorname{cor}_{R/I}^R : G(R/I) \to G(R)$$

   defined in Exercise 1.4.4 is an isomorphism. Compare with Exercise 1.2.8. (Hint: for any R-module M, the quotients $I^kM/I^{k+1}M$ have the structure of an R/I-module; consider the function $G(R) \to G(R/I)$ that sends a class $[M]$ to $\sum_{k\geq 0}[I^kM/I^{k+1}M]$).
   (b) Use part (a) to prove a converse to Exercise 1.4.1 for a Noetherian ring R, i.e. show that a Noetherian ring R has the property $G(R) = 0$ only if $R = 0$.

7. (*Gysin morphism*). Let R be a ring and let $f \in R$ be any element of R which is not a zero-divisor in R. Let $\pi : R \to R/fR$ be the canonical projection map. In this exercise we construct a group homomorphism

$$\pi^! : G(R) \to G(R/fR) \qquad [M] \mapsto [M/fM] - [\operatorname{Tor}_1^R(R/fR, M)]$$

called *the Gysin morphism of* π.
   (a) Let M be an R-module and let $I \subset R$ be an ideal. Show that $\operatorname{Tor}_1^R(R/I, M)$ is isomorphic with the kernel of the R-module homomorphism

   $$I \otimes_R M \to M \qquad f \otimes m \mapsto fm.$$

   Conclude that $\operatorname{Tor}_1^R(R/fR, M)$ is isomorphic with the R-submodule $M\{f\}$ of M consisting of all elements $m \in M$ such that $fm = 0$ in M.
   (b) Let L, M, N be three R-modules fitting in an exact sequence

   $$0 \to L \to M \to N \to 0.$$

   Prove that there is a long exact sequence

   $$0 \to L\{f\} \to M\{f\} \to N\{f\} \to L/fL \to M/fM \to N/fN \to 0.$$

(c) Define a map

$$M_{fg}(R) \to G(R/fR) \qquad M \mapsto [M/fM] - [M\{f\}].$$

Use parts (a) and (b) of this exercise to show that this descends to give the map $\pi^! : G(R) \to G(R/fR)$ defined above.

8. (*Homotopy invariance of G-theory*). Let R be a Noetherian ring and let $R[x]$ be the ring of polynomials in the single variable x over R. We will show that the map

$$\mathrm{res}_R^{R[x]} : G(R) \to G(R[x])$$

of Exercise 1.4.4 is an isomorphism with inverse the map $\pi^! : G(R[x]) \to G(R)$ of Exercise 1.4.7 associated to $\pi : R[x] \to R$ which sends x to 0.

(a) Prove that the composition

$$G(R) \xrightarrow{\mathrm{res}_R^{R[x]}} G(R[x]) \xrightarrow{\pi^!} G(R)$$

is the identity on $G(R)$. Hence the map $\mathrm{res}_R^{R[x]}$ is injective.

(b) Consider the set of prime ideals of R defined as

$$\mathcal{P} = \{\mathfrak{q} \cap R : \mathfrak{q} \in \mathrm{Spec}(R[x]) \text{ is such that } [R[x]/\mathfrak{q}] \notin \mathrm{Im}(\mathrm{res}_R^{R[x]})\}.$$

Then $\mathrm{res}_R^{R[x]}$ is surjective if and only if $\mathcal{P} = \emptyset$. Suppose, for a contradiction, that $\mathcal{P} \neq \emptyset$. Then, since R is Noetherian, there is a maximal element $\mathfrak{p} \in \mathcal{P}$ with regards to containment.

Let $\mathfrak{p} = \mathfrak{q} \cap R$. Mimic the strategy from (b) of Exercise 1.3.11 to argue $\mathfrak{q} \supsetneq \mathfrak{p}R[x]$ and prove that there is an element $g \in \mathfrak{q}$ such that

$$\mathfrak{q} = \{f \in R[x] : \text{there exists } a \in R \setminus \mathfrak{p} \text{ so that } af \in (g) + \mathfrak{p}R[x]\}.$$

(c) Now there are exact sequences

$$0 \to R[x]/\mathfrak{p}R[x] \xrightarrow{\cdot g} R[x]/\mathfrak{p}R[x] \to R[x]/(\mathfrak{p}R[x] + (g)) \to 0$$

and

$$0 \to \ker(\varphi) \to R[x]/(\mathfrak{p}R[x] + (g)) \xrightarrow{\varphi} R[x]/\mathfrak{q} \to 0.$$

Prove that φ becomes an isomorphism after localizing at the multiplicative set $R \setminus \mathfrak{p}$. Since $\ker(\varphi)$ is finitely generated, this implies that there is an element $h \in R \setminus \mathfrak{p}$ so that $h \cdot \ker(\varphi) = 0$. By considering a prime filtration of $\ker(\varphi)$, prove that $[R[x]/\mathfrak{q}]$ is in the image of $\mathrm{res}_R^{R[x]}$. As this contradicts the assumption that $\mathfrak{p} \in \mathcal{P}$, we must have $\mathcal{P} = \emptyset$.

9. Let k be a fixed algebraically closed field and set $R = k[x,y]/(y^2-x^3)$. Following Exercise 1.2.10, identify R with $k[t^2,t^3] \subset k[t]$ by the map $f : R \to k[t]$ which sends $f(x) = t^2$ and $f(y) = t^3$. The ring $k[t]$, considered as an R-module, is generated by 1 and t so any finitely generated $k[t]$-module is also a finitely generated R-module with respect to the map f.

   Consider the induced map

$$\mathrm{cor}^R_{k[t]} : G(k[t]) \to G(R) \quad \text{where} \quad [M] \mapsto [M]$$

   which considers any $k[t]$-module as an R-module with respect to f. Prove that $\mathrm{cor}^R_{k[t]}$ is surjective and prove that there is an isomorphism $G(R) \cong \mathbb{Z}$.

1.5 Regular rings

Let $(R,\mathfrak{m})$ be a Noetherian local ring. Then $\mathfrak{m}$ has finite height, see Remark 1.4.4, which is necessarily also equal to the Krull dimension of R, i.e. $\mathrm{Kr.dim}(R) = \mathrm{ht}(\mathfrak{m})$. If $(x_1,...,x_r) = \mathfrak{m}$ is a minimal set of generators for the maximal ideal of R, then by Nakayama's Lemma [AM69, Proposition 2.8] we have $\dim_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) = r$. Hence, by Remark 1.4.4 again, there is an inequality

$$\mathrm{Kr.dim}(R) \leq \dim_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) = r. \tag{1.5.1}$$

We say that $(R,\mathfrak{m})$ is a *regular local ring* if the inequality in (1.5.1) is an equality.

This is related to a descending filtration $F^\bullet_\mathfrak{m}(R)$ on R induced by taking higher powers of $\mathfrak{m}$, i.e. for any $i \geq 0$ we have $F^i_\mathfrak{m}(R) = \mathfrak{m}^i$; here $\mathfrak{m}^0$ is defined to be R. Associated to this filtration is a graded group $\mathrm{gr}_\mathfrak{m}(R)$ defined as

$$\mathrm{gr}_\mathfrak{m}(R) = \bigoplus_{i\geq 0} \mathrm{gr}^i_\mathfrak{m}(R) \quad \text{where} \quad \mathrm{gr}^i_\mathfrak{m}(R) = F^i_\mathfrak{m}(R)/F^{i+1}_\mathfrak{m}(R) = \mathfrak{m}^i/\mathfrak{m}^{i+1}.$$

The group $\mathrm{gr}_\mathfrak{m}(R)$ can further be given the structure of a graded ring with the following multiplication: if $x \in \mathfrak{m}^i$ and $y \in \mathfrak{m}^j$ are elements with images $\bar{x}, \bar{y}$ in $\mathrm{gr}^i_\mathfrak{m}(R)$ and $\mathrm{gr}^j_\mathfrak{m}(R)$ respectively, then the product $\bar{x}\cdot\bar{y}$ is defined as the image $\overline{xy}$ of xy inside $\mathrm{gr}^{i+j}_\mathfrak{m}(R)$; this is well defined since if x' is another element with image $\bar{x}$ in $\mathrm{gr}^i_\mathfrak{m}(R)$ then $x - x'$ is contained in $\mathfrak{m}^{i+1}$.

An equivalent condition for a Noetherian local ring $(R,\mathfrak{m})$ to be regular is then that there is an isomorphism $\mathrm{gr}_\mathfrak{m}(R) \cong (R/\mathfrak{m})[t_1,...,t_d]$ with the polynomial ring over $R/\mathfrak{m}$ in $d = \mathrm{Kr.dim}(R)$ many independent variables [AM69, Theorem 11.22]. Consequently, this result implies that a regular local ring R is an integral domain: if $x \in \mathfrak{m}^i \setminus \mathfrak{m}^{i+1}$ and $y \in \mathfrak{m}^j \setminus \mathfrak{m}^{j+1}$ are nonzero elements with nonzero images $\bar{x},\bar{y}$ in $\mathrm{gr}^i_\mathfrak{m}(R)$ and $\mathrm{gr}^j_\mathfrak{m}(R)$, then $\overline{xy} \neq 0$ in $\mathrm{gr}^{i+j}_\mathfrak{m}(R)$ so that $xy \neq 0$ in $\mathfrak{m}^{i+j} \subset R$.

Definition 1.5.2. Let R be a commutative ring. We say that R is a *regular* ring if, for every prime ideal $\mathfrak{p} \subset R$, the local ring $R_{\mathfrak{p}}$ is a regular local ring.

Our interest in regular rings comes from the following theorem which says that, for a regular ring R, the groups $K(R)$ and $G(R)$ are canonically isomorphic.

Theorem 1.5.3. *Let R be ring. Write*

$$\varphi_R : K(R) \to G(R) \qquad [P] \mapsto [P]$$

for the group homomorphism induced by the canonical inclusion $P_{fg}(R) \subset M_{fg}(R)$. If R is a regular ring with $\mathrm{Kr.dim}(R) < \infty$, then φ_R is an isomorphism.

We will prove Theorem 1.5.3 much later in this section, after a healthy amount of effort is devoted to developing both the homological and algebraic properties of regular local rings. For now, in order to give context to their definition, we make some immediate remarks on regular local rings and their global counterparts.

Remark 1.5.4. For a ring R, the property of being regular is intimately connected with the lack of singularities of the affine scheme $\mathrm{Spec}(R)$. To be precise, recall if R is a finitely generated k-algebra, for a field k, then the tangent space to a point $\mathfrak{p} \in \mathrm{Spec}(R)$ is the $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$-vector space

$$T_{\mathrm{Spec}(R),\mathfrak{p}} = \mathrm{Hom}_{R_{\mathfrak{p}}}(\Omega_{R/k,\mathfrak{p}}, R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}})$$

where $\Omega_{R/k}$ is the R-module of Kähler differentials of R over k. When $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ is a separable field extension of k there is also an isomorphism $\Omega_{R/k,\mathfrak{p}} \cong \mathfrak{p}R_{\mathfrak{p}}/\mathfrak{p}^2R_{\mathfrak{p}}$ by [Sta19, Tag 0B2E]. Combining these, we get that the tangent space at a point $\mathfrak{p} \in \mathrm{Spec}(R)$ has the same dimension as the local scheme $\mathrm{Spec}(R_{\mathfrak{p}})$ if $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ is separable over k and $R_{\mathfrak{p}}$ is regular. The converse also holds if the field k is perfect, e.g. if the characteristic of k is zero or if k is a finite field.

Remark 1.5.5. If R is a finitely generated k-algebra, for a field k, and if $\mathfrak{m} \subset R$ is a maximal ideal of R with $R/\mathfrak{m} \cong k$, then the graded ring $\mathrm{gr}_{\mathfrak{m}R_{\mathfrak{m}}}(R_{\mathfrak{m}})$ is the coordinate ring of *the tangent cone to* $\mathrm{Spec}(R)$ *at the point* $\mathfrak{m}$.

As a special case, suppose that there is a surjective ring map

$$\phi : k[x_1, ..., x_n] \to R$$

realizing $\mathrm{Spec}(R) \subset \mathbb{A}^n_k$ as a closed subscheme passing through the origin of $\mathbb{A}^n_k$, so there is a maximal ideal $\mathfrak{m} \subset R$ with $\phi^{-1}(\mathfrak{m}) = (x_1, ..., x_n)$. The graded ring $\mathrm{gr}_{\mathfrak{m}R_{\mathfrak{m}}}(R_{\mathfrak{m}})$ then admits a map

$$\phi' : k[x_1, ..., x_n] \to \mathrm{gr}_{\mathfrak{m}R_{\mathfrak{m}}}(R_{\mathfrak{m}}) \cong \bigoplus_{i \geq 0} \mathfrak{m}^i/\mathfrak{m}^{i+1} \qquad x_j \mapsto \phi(x_j)$$

which realizes $\mathrm{Spec}(\mathrm{gr}_{\mathfrak{m}R_\mathfrak{m}}(R_\mathfrak{m}))$ as a closed subscheme of $\mathbb{A}^n_k$ also passing through the origin. If R is an integral domain, then the Krull dimensions of these schemes are all the same

$$\mathrm{Kr.dim}(R) = \mathrm{Kr.dim}(R_\mathfrak{m}) = \mathrm{Kr.dim}(\mathrm{gr}_{\mathfrak{m}R_\mathfrak{m}}(R_\mathfrak{m}));$$

the first equality on the left, which requires the integral domain hypothesis, holds by [AM69, Theorem 11.25] and the second, which is true more generally without the integral domain hypothesis on R, by [Eis95, Corollary 12.5].

Example 1.5.6. If R is a Noetherian ring having $\mathrm{Kr.dim}(R) = 0$, then R is regular if and only if R is a finite product of fields. In general, if R and S are two regular rings, then $R \times S$ is again a regular ring and, conversely, if a product $R \times S$ is regular then both R and S are regular too.

Suppose now that R is a regular Noetherian ring with $\mathrm{Kr.dim}(R) = 1$. If we decompose R into a product of rings so that each of the factors has connected spectrum then we could write

$$R \cong R_1 \times \cdots \times R_n \times F_1 \times \cdots \times F_m$$

for regular rings R_i, with $\mathrm{Kr.dim}(R_i) = 1$, and fields F_j. Each ring R_i must have a unique minimal prime ideal since if two distinct minimal prime ideals $\mathfrak{p}, \mathfrak{q}$ existed in R_i there would be a maximal ideal $\mathfrak{m}$ containing both $\mathfrak{p}, \mathfrak{q}$ which implies that the localization $R_\mathfrak{m}$ is not a domain [AM69, Proposition 4.7].

Now each of the rings R_i, over the varying $1 \leq i \leq n$, is necessarily a domain: since R_i is regular we know that R_i is reduced, as this holds locally; if $\mathfrak{p}_i$ is the unique minimal prime ideal for R_i, it follows $\mathfrak{p}_i$ is the set of all nilpotent elements of R_i, i.e. $\mathfrak{p}_i = (0)$. Also, if $\mathfrak{m} \subset R_i$ is any maximal ideal then $\dim_{R_i/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) = 1$. This latter condition is equivalent to the localization $(R_i)_\mathfrak{m}$ being a DVR [AM69, Proposition 9.2] and, hence, R_i is integrally closed. Thus a Noetherian ring R with $\mathrm{Kr.dim}(R) = 1$ is regular if and only if R is the finite product of some Dedekind domains and fields.

Proposition 1.5.7. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and suppose that $\mathrm{gr}_\mathfrak{m}(R)$ is an integrally closed domain. Then R is integrally closed as well. Therefore any regular local ring, and any regular integral domain, is integrally closed.*

Proof. Assuming that $\mathrm{gr}_\mathfrak{m}(R)$ is an integral domain, it follows that R is also an integral domain using the same argument as above Definition 1.5.2. Let $f = g/h$ be an element in the fraction field $F = R_{(0)}$ of R and suppose that f is integral. We need to show that f is contained in R and, to do this, we'll show that g is contained in the ideal $(h) \subset R$.

More precisely, we're going to show that g is an element of the ideal $(h) + \mathfrak{m}^i$ for all $i \geq 0$. It then follows that

$$g \in \bigcap_{i \geq 0} \left((h) + \mathfrak{m}^i \right) = (h). \tag{1.5.8}$$

The equality of ideals in the above is clear if $(h) = R$ and otherwise comes from [AM69, Corollary 10.19] which says that the intersection of all nonnegative powers of the maximal ideal in $R/(h)$ is equal the zero ideal of this quotient. The proof now boils down to the following claim:

Lemma 1.5.9. *Let $(R, \mathfrak{m})$ be any Noetherian local ring with $\mathrm{gr}_{\mathfrak{m}}(R)$ an integrally closed domain. Let f be an element of the fraction field $F = R_{(0)}$ which is integral over R and write $f = g/h$ for some $g, h \in R$.*

In this setting, if there exists an element $\mu \in R$ so that $h(f - \mu) \in \mathfrak{m}^i$, for some integer $i \geq 0$, then there exists an element $\mu' \in R$ so that $h(f - \mu') \in \mathfrak{m}^{i+1}$.

Proof. Since f is integral over R, there is a relation

$$f^n + r_{n-1} f^{n-1} + \cdots + r_0 = 0, \quad \text{for some } r_0, ..., r_{n-1} \in R,$$

and for some $n \geq 1$. Setting $c = h^{n-1}$ we have $cf^m \in R$ for all $m \geq 1$. If μ is given, then similarly $c(f - \mu)^m \in R$ for all $m \geq 1$ since we can expand this expression using the binomial theorem.

For any $m \geq 1$, multiplying $c(f - \mu)^m$ by h^m gives

$$h^m \cdot c(f - \mu)^m = c\left(h(f - \mu)\right)^m = c(g - h\mu)^m. \tag{1.5.10}$$

Let $j \geq 0$ be such that $c \in \mathfrak{m}^j \setminus \mathfrak{m}^{j+1}$ and let $\bar{c} \neq 0$ denote the image of c in $\mathrm{gr}^j_{\mathfrak{m}}(R)$. Similarly, let k be such that $h \in \mathfrak{m}^k \setminus \mathfrak{m}^{k+1}$ and $\bar{h} \neq 0$ the image of h in $\mathrm{gr}^k_{\mathfrak{m}}(R)$. We write $\omega \in \mathrm{gr}^i_{\mathfrak{m}}(R)$ for the image of $g - h\mu = h(f - \mu) \in \mathfrak{m}^i$.

Fix an integer $m \geq 1$ and let $l = m(i - k) + j$. One might expect from (1.5.10) that the element $c(f - \mu)^m$ is contained in $\mathfrak{m}^l$. This is the case, as we now show. Suppose, on the contrary, that $c(f - \mu)^m$ isn't contained in $\mathfrak{m}^l$. Then $c(f - \mu)^m$ is necessarily contained in $\mathfrak{m}^s \setminus \mathfrak{m}^{s+1}$ for some $0 \leq s < l$. If we write γ_m for the image of $c(f - \mu)^m$ in $\mathrm{gr}^s_{\mathfrak{m}}(R)$, then $\bar{h}^m \gamma_m \neq 0$ since $\mathrm{gr}_{\mathfrak{m}}(R)$ is an integral domain. However, the element $\bar{h}^m \gamma_m$ is also the image of $c(g - h\mu)^m$ in $\mathrm{gr}^{km+s}_{\mathfrak{m}}(R)$, because of the equality (1.5.10), which must be 0 since $c(g - h\mu)^m$ is contained in $\mathfrak{m}^{km+l}$ and $km + l \geq km + s + 1$; this is a contradiction.

For every $m \geq 1$, we can therefore set γ_m to be the image of $c(f - \mu)^m$ in $\mathrm{gr}^l_{\mathfrak{m}}(R)$ where $l = m(i - k) + j$. The equality in (1.5.10) then shows that $\bar{h}^m \gamma_m = \bar{c}\omega^m$. Hence the element $\omega/\bar{h}$ of the fraction field of $\mathrm{gr}_{\mathfrak{m}}(R)$ has the property that $\bar{c}(\omega/\bar{h})^m$ is contained in $\mathrm{gr}_{\mathfrak{m}}(R)$ for all $m \geq 1$. Since $\bar{c} \neq 0$, this implies that $\omega/\bar{h}$ is an integral element over $\mathrm{gr}_{\mathfrak{m}}(R)$. Indeed, since $\bar{c}(\omega/\bar{h})^m$ is contained in $\mathrm{gr}_{\mathfrak{m}}(R)$ for all $m \geq 1$, we have a containment

$$\mathrm{gr}_{\mathfrak{m}}(R)[\omega/\bar{h}] \subset (1/\bar{c})\mathrm{gr}_{\mathfrak{m}}(R)$$

of $\mathrm{gr}_{\mathfrak{m}}(R)$-submodules of the fraction field of $\mathrm{gr}_{\mathfrak{m}}(R)$. Since $\mathrm{gr}_{\mathfrak{m}}(R)$ is Noetherian, by [AM69, Proposition 10.22], this implies that $\mathrm{gr}_{\mathfrak{m}}(R)[\omega/\bar{h}]$ is finitely generated. Hence $\omega/\bar{h}$ is integral over $\mathrm{gr}_{\mathfrak{m}}(R)$ by [AM69, Proposition 5.1].

Now, as $\mathrm{gr}_{\mathfrak{m}}(R)$ is integrally closed, the element $\omega/\bar{h}$ is contained in $\mathrm{gr}^{i-k}_{\mathfrak{m}}(R)$. Thus there exists an element $r \in \mathfrak{m}^{i-k}$ with image $\omega/\bar{h}$ in $\mathrm{gr}^{i-k}_{\mathfrak{m}}(R)$ which satisfies $g - h\mu - rh \in \mathfrak{m}^{i+1}$. Setting $\mu' = \mu - r$ completes the proof. □

To finish the proof for Proposition 1.5.7, we set $\mu = 0$. Then $h(f-\mu) = hf = g$ is contained in $\mathfrak{m}^i$ for some $i \geq 0$ so that g is also contained in $(h) + \mathfrak{m}^j$ for all $0 \leq j \leq i$. By Lemma 1.5.9, we can find an element μ' in R so that

$$h(f - \mu') = hf - h\mu' = g - h\mu'$$

is contained in $\mathfrak{m}^{i+1}$. Hence $g \in (h)+\mathfrak{m}^{i+1}$. Applying Lemma 1.5.9 again, this time using μ' as our starting point, we can find an element μ'' so that $h(f-\mu'') = g-h\mu''$ is contained in $\mathfrak{m}^{i+2}$. Hence $g \in (h) + \mathfrak{m}^{i+2}$. Continuing in this way, we find that the containment of (1.5.8) is satisfied so that f is an element of R as desired. □

Our goal now is to prove Theorem 1.5.3 but, before we do this, we're going to develop the homological theory of regular local rings from scratch. The main result that we need, in this direction, is a fundamental theorem, due to Auslander, Buchsbaum and Serre, which shows that a Noetherian local ring is regular if and only if every finitely generated module admits a finite resolution by free modules. Once the local theory has been built, we'll show how one can reduce some problems in homological algebra for regular rings from the global case to the local case. This will finish our preparation for the proof of Theorem 1.5.3.

Along the way to proving Theorem 1.5.3, we show how the cumulative theory that we've developed so far in this chapter can be used to obtain nontrivial results on the structure of regular rings. Namely, we show how one can use the results of Sections 1.3 and 1.4 to prove that regular rings are locally factorial, a fact that we alluded to in Section 1.3. Independently of this, we also determine the structure of the K-theory of a Dedekind domain and show how the isomorphism of Theorem 1.5.3 can be viewed as a natural extension of the isomorphism from Theorem 1.3.27 between the Picard group and the divisor class group for these rings.

Regular local rings

At the moment, let R be any ring. If M is an R-module, then we say that M admits a *resolution by finitely generated projective R-modules* if there exists a collection

$\{P_i\}_{i\geq 0}$ with each P_i a finitely generated and projective R-module together with an exact sequence (possibly infinite):

$$\cdots \to P_{i+1} \to P_i \to \cdots \to P_1 \to P_0 \to M \to 0.$$

We say that M admits a *finite resolution by finitely generated projective R-modules* if there exists such a resolution with the property that there is an integer $N \geq 0$ so that $P_n = 0$ for all $n > N$. When there's no risk of being confusing, we'll call such a resolution simply a *finite resolution*. A finite resolution of an R-module M is said to have *length* $N \geq 0$ if $P_N \neq 0$ and $P_n = 0$ for all $n > N$.

Definition 1.5.11. Let R be a ring and let $M \neq 0$ be an R-module. If M admits a finite resolution by finitely generated projective R-modules, then the *projective dimension* of M is defined as the number

$$\mathrm{pd}_R(M) := \min\{n \in \mathbb{Z}_{\geq 0} : \text{there exists a finite resolution of } M \text{ with length } n\}.$$

If M does not admit a finite resolution, then we say that M has infinite projective dimension and we write $\mathrm{pd}_R(M) = \infty$. By convention, we set $\mathrm{pd}_R(0) = -1$.

The following theorem is often called the Auslander-Buchsbaum Theorem.[1]

Theorem 1.5.12. *Let $(R, \mathfrak{m})$ be a Noetherian local ring. The following conditions are then equivalent for the ring R:*
(1) *R is a regular local ring,*
(2) *each finitely generated R-module M has finite projective dimension,*
(3) $\mathrm{pd}_R(R/\mathfrak{m}) < \infty$.

Proving this theorem will take some work. The equivalence between the above is gotten by showing that (1) implies (2) implies (3) implies (1). However, condition (2) of Theorem 1.5.12 clearly implies (3) since $R/\mathfrak{m}$ is a finitely generated R-module. This means that there are only two remaining implications that need to be justified. We'll prove (1) implies (2) first since this will give us an opportunity to introduce some more terminology. Then, and this is the more difficult part of the proof, we'll show that condition (3) implies (1).

For any ring R and for any R-module M, we can always construct a resolution of M by free R-modules of possibly infinite rank. To do this, one can choose a generating set $\{x_i\}_{i\in I_0}$ for the R-module M indexed by a set I_0. This corresponds to a surjection $\phi_0 : R^{\oplus I_0} \to M$ defined by sending the basis element e_i of $R^{\oplus I_0}$ to x_i in M. This will be the first morphism in such a resolution. To get the next term in the resolution, one chooses a generating set for the kernel of ϕ_0, say $\{x_i\}_{i\in I_1}$,

[1]For some history on the problem from the perspective of David Buchsbaum, see [Buc].

indexed by a set I_1. There's then a similarly defined map $\phi_1 : R^{\oplus I_1} \to R^{\oplus I_0}$ with image exactly the kernel of ϕ_0. Repeating this procedure yields a, possibly infinite, exact sequence

$$\cdots \to R^{\oplus I_{j+1}} \xrightarrow{\phi_{j+1}} R^{\oplus I_j} \to \cdots \to R^{\oplus I_0} \xrightarrow{\phi_0} M \to 0. \tag{1.5.13}$$

Any exact sequence that's constructed in this way will be called a *free resolution* of the R-module M.

Now we specialize to the case $(R, \mathfrak{m})$ is a local ring. In this case, any resolution of an R-module M by finitely generated projective R-modules is a free resolution.

Definition 1.5.14. Let $(R, \mathfrak{m})$ be a local ring and let M be an R-module with a free resolution

$$\cdots \to R^{\oplus I_{j+1}} \xrightarrow{\phi_{j+1}} R^{\oplus I_j} \to \cdots \to R^{\oplus I_0} \xrightarrow{\phi_0} M \to 0.$$

We say that this free resolution is a *minimal free resolution of* M if, for each $j > 0$, one has $\phi_j \otimes \mathrm{id}_{R/\mathfrak{m}} = 0$ for the morphism obtained by tensoring ϕ_j with $R/\mathfrak{m}$.

In order to construct a minimal free resolution for an R-module M over $(R, \mathfrak{m})$, it suffices, in the construction of a free resolution like (1.5.13), to choose for each $j \geq 0$ a minimal set of generators indexed by the set I_j. Indeed, if $\{x_i\}_{i \in I_j}$ is a minimal set of generators from the jth step in this construction, then the morphism ϕ_j has the property that $\phi_j(R^{\oplus I_j}) \subset \mathfrak{m}R^{\oplus I_{j-1}}$ whenever $j > 0$. To see this, let $\{x_i\}_{i \in I_{j-1}}$ be minimal and consider the exact sequence

$$R^{\oplus I_{j+1}} \xrightarrow{\phi_{j+1}} R^{\oplus I_j} \xrightarrow{\phi_j} R^{\oplus I_{j-1}}.$$

An element $\sum_{s=1}^m r_{i_s} e_{i_s}$, with $r_i \in R$ and with the e_{i_s} basis elements of $R^{\oplus I_j}$, is in the image of ϕ_{j+1} if and only if $\sum_{s=1}^m r_{i_s} x_{i_s} = 0$ in $R^{\oplus I_{j-1}}$. If we assume, without loss of generality, that the element r_{i_1} was contained in $R \setminus \mathfrak{m}$ then r_{i_1} would be a unit and we could write

$$x_{i_1} = -\frac{1}{r_{i_1}} \left(\sum_{s=2}^m r_{i_s} x_{i_s} \right)$$

inside $R^{\oplus I_{j-1}}$. As this contradicts the minimality of $\{x_i\}_{i \in I_{j-1}}$, we must have that all the coefficients r_{i_s} are contained in $\mathfrak{m}$ for all $1 \leq s \leq m$.

Remark 1.5.15. If $(R, \mathfrak{m})$ is an arbitrary local ring and if M is an R-module, then it's not obvious that there exists a minimal generating set for M. If M is finitely generated, however, then the existence of a minimal generating set is immediate. So, if $(R, \mathfrak{m})$ is a Noetherian local ring and if M is a finitely generated R-module then, by repeated use of part (2) of Lemma 1.1.7, a minimal free resolution of M (by free R-modules of finite rank) exists.

Proposition 1.5.16. *Let $(R, \mathfrak{m})$ be a local Noetherian ring and let $M \neq 0$ be a finitely generated R-module. Then the following are either all infinite or all finite:*

(1) *the length of any (and hence every) minimal free resolution of M*
(2) *the supremum $\sup\{n \in \mathbb{Z}_{\geq 0} : \operatorname{Tor}_n^R(M, R/\mathfrak{m}) \neq 0\}$*
(3) *the projective dimension $\operatorname{pd}_R(M)$*

Moreover, if any of the above is finite then all three of these numbers are all equal.

Proof. Pick one minimal free resolution of M, say

$$\cdots \to P_i \to \cdots \to P_0 \to M \to 0.$$

As M is finitely generated and R is Noetherian, each of the free R-modules P_i has finite rank. Tensoring with $R/\mathfrak{m}$ and omitting the rightmost term gives a sequence

$$\cdots \to P_i \otimes_R R/\mathfrak{m} \to \cdots \to P_0 \otimes_R R/\mathfrak{m} \to 0$$

where each of the terms $P_i \otimes_R R/\mathfrak{m}$ is a finite dimensional $R/\mathfrak{m}$-vector space with dimension equal to the rank of the R-module P_i and with all maps zero. Moreover, the homology in the ith spot of this last sequence is exactly $\operatorname{Tor}_i^R(M, R/\mathfrak{m})$.

By Nakayama's Lemma [AM69, Proposition 2.8] a finitely generated R-module P satisfies $P \otimes_R R/\mathfrak{m} \cong P/\mathfrak{m}P = 0$ if and only if $P = 0$. So, there exists an $N \geq 0$ so that $\operatorname{Tor}_n^R(M, R/\mathfrak{m}) = 0$ for all $n > N$ if, and only if, in any minimal free resolution for M, the free R-module in the nth place is zero. This proves that either (1) and (2) are both finite and equal or both (1) and (2) are infinite.

If (3) is infinite, then a minimal free resolution for M can not have finite length. Hence (2) is also infinite in this case. Conversely, if (2) is infinite then M can not admit any projective resolution of finite length as otherwise $\operatorname{Tor}_n^R(M, R/\mathfrak{m}) = 0$ for all sufficiently large n. Similarly, if (2) is finite and equal to some number N, then a minimal free resolution of M must have length N and there can be no shorter length resolution of M since $\operatorname{Tor}_n^R(M, R/\mathfrak{m}) \neq 0$. Hence $\operatorname{pd}_R(M) = N$.

Finally, if (3) is finite and $\operatorname{pd}_R(M) = N$, then $\operatorname{Tor}_n^R(M, R/\mathfrak{m}) = 0$ for all $n > N$. Thus, every minimal free resolution for M must have length less or equal to N. But this means that every minimal free resolution for M has the same length, equal to N, and this implies that (2) is also equal to N. $\square$

Corollary 1.5.17. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and let L, M, and N be finitely generated R-modules fitting into an exact sequence*

$$0 \to L \to M \to N \to 0.$$

If any two of the R-modules L, M, N have finite projective dimension, then all three of the R-modules L, M, N have finite projective dimension.

Moreover, if both $\operatorname{pd}_R(M)$ and $\operatorname{pd}_R(N)$ are finite and if $\operatorname{pd}_R(M) < \operatorname{pd}_R(N)$, then $\operatorname{pd}_R(L) = \operatorname{pd}_R(N) - 1$.

Proof. For the first claim, notice that there is a long exact sequence

$$\begin{tikzcd}[column sep=small] \cdots \arrow[r] & \mathrm{Tor}_{i+1}^R(N, R/\mathfrak{m}) \arrow[r] & \mathrm{Tor}_i^R(L, R/\mathfrak{m}) \arrow[r] & \mathrm{Tor}_i^R(M, R/\mathfrak{m}) \arrow[dll, out=-10, in=170] & \\ & \mathrm{Tor}_i^R(N, R/\mathfrak{m}) \arrow[r] & \mathrm{Tor}_{i-1}^R(L, R/\mathfrak{m}) \arrow[r] & \mathrm{Tor}_{i-1}^R(M, R/\mathfrak{m}) \arrow[r] & \cdots \end{tikzcd}$$

associated with the given short exact sequence of L, M, and N. By the equivalence of (2) and (3) from Proposition 1.5.16, if there exists a number $n \geq 1$ so that, e.g.

$$\mathrm{Tor}_i^R(L, R/\mathfrak{m}) = \mathrm{Tor}_i^R(M, R/\mathfrak{m}) = 0$$

for all $i \geq n$, then $\mathrm{Tor}_i^R(N, R/\mathfrak{m}) = 0$ for all $i \geq n$ as well.

For the second claim, suppose that $\mathrm{pd}_R(M) < \mathrm{pd}_R(N) < \infty$. Let $\mathrm{pd}_R(N) = n$. Then, from the same long exact as above, we have that $\mathrm{Tor}_i^R(L, R/\mathfrak{m}) = 0$ for all $i \geq n$ and there is an exact sequence

$$0 = \mathrm{Tor}_n^R(M, R/\mathfrak{m}) \to \mathrm{Tor}_n^R(N, R/\mathfrak{m}) \to \mathrm{Tor}_{n-1}^R(L, R/\mathfrak{m}).$$

Since this means that $\mathrm{Tor}_n^R(N, R/\mathfrak{m}) \neq 0$ injects into $\mathrm{Tor}_{n-1}^R(L, R/\mathfrak{m})$, it follows that $\mathrm{pd}_R(L) = n - 1$. $\square$

The proofs for both of the implications (1) implies (2) and (3) implies (1) of Theorem 1.5.12 rely on the following observation for regular local rings that allows one to use induction on the dimension of the ring.

Lemma 1.5.18. *Let $(R, \mathfrak{m})$ be a regular local ring and $I \subset R$ be a proper ideal. Then the local ring R/I is regular if and only if there exists a minimal set of generators S for $\mathfrak{m}$ such that I is generated by a subset $S_0 \subset S$.*

Proof. If $\mathfrak{m} = 0$ then R is a field, and all is clear. So assume that $\mathrm{Kr.dim}(R) \geq 1$. Then, for one direction, assume that I is generated by a subset $S_0 \subset S$ of a minimal set of generators $S = \{x_1, ..., x_d\}$ for $\mathfrak{m}$. If S_0 is empty, then $I = 0$ and R is regular by assumption. Otherwise, there is some $x \in S_0$. Since $x \in \mathfrak{m}$ we have that x is a nonunit, non-zero divisor. Thus $R/(x)$ is a local ring, with maximal ideal $\mathfrak{m}_1 = \mathfrak{m}/(x)$, satisfying

$$\mathrm{Kr.dim}(R/(x)) = \mathrm{Kr.dim}(R) - 1 \quad \text{and} \quad \dim_{R/\mathfrak{m}}(\mathfrak{m}_1/\mathfrak{m}_1^2) = \dim_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) - 1$$

by [AM69, Corollary 11.18]. Hence $R/(x)$ is a regular local ring. Repeating this argument with $S_0 \backslash \{x\}$ and $R/(x)$, and so on, shows that R/I is regular as claimed.

For the other direction, we go by induction: assume that if R is a regular local ring with $\mathrm{Kr.dim}(R) < d$, then an ideal $I \subset R$ is such that R/I is regular if and only if I is generated by a subset of a minimal set of generators for $\mathfrak{m}$. Let R be a regular ring with $\operatorname{Kr.dim}(R) = d$, let $I \subset R$ be a nonzero ideal with R/I regular, and write $\mathfrak{m}_1 = \mathfrak{m}/I$. Choose an element $x \in I$ with $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ (at least one such element exists since if $I \subset \mathfrak{m}^2$ then

$$\dim_{R/\mathfrak{m}}(\mathfrak{m}_1/\mathfrak{m}_1^2) = \dim_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2)$$

but $\operatorname{Kr.dim}(R/I) < \operatorname{Kr.dim}(R)$ by Remark 1.4.4). Since $\operatorname{Kr.dim}(R) = d$, we have that $\operatorname{Kr.dim}(R/(x)) = d - 1$ by [AM69, Corollary 11.18] so that $R/(x)$ is regular. By our induction hypothesis the ideal $I/(x)$ is generated by a set of elements S_0' which forms a subset of a minimal set of generators S' for $\mathfrak{m}/(x)$. If S^* denotes a set of lifts to $\mathfrak{m}$ from the elements of the set S', then by $S = S^* \cup \{x\}$ is a minimal generating set for $\mathfrak{m}$ as desired. $\square$

Proof of (1) $\implies$ (2). To illustrate the type of inductive argument that we can now use, we'll start the proof that if $(R, \mathfrak{m})$ is a regular local ring and M is a finitely generated R-module, then $\operatorname{pd}_R(M)$ is finite. We'll need to prove one lemma in the middle of the proof of this implication but, it's easier to understand why the lemma is being proved after seeing the argument for this implication.

We go by induction on the Krull dimension of R. If $\operatorname{Kr.dim}(R) = 0$, then R is a field and M is a finite dimensional vector space over R; hence $\operatorname{pd}_R(M) = 0$. For our induction assumption, we suppose that for every regular local ring R with $\operatorname{Kr.dim}(R) < d$, and for any finitely generated R-module M, we have an inequality $\operatorname{pd}_R(M) \leq \operatorname{Kr.dim}(R)$. For the induction step we assume that $(R, \mathfrak{m})$ is a regular local ring, M is a finitely generated R-module, $\operatorname{Kr.dim}(R) = d$, and we'll show that $\operatorname{pd}_R(M) \leq d$. Since M is finitely generated, there is a surjection $\phi : R^{\oplus n} \to M$ for some integer $n \geq 0$. We denote by $N = \ker(\phi)$ the kernel of this map.

From the short exact sequence

$$0 \to N \to R^{\oplus n} \xrightarrow{\phi} M \to 0$$

we get a long exact sequence of $\operatorname{Tor}_*^R(-, R/\mathfrak{m})$ that looks like:

$$\begin{tikzcd}
\cdots \arrow[r] & \operatorname{Tor}_{i+1}^R(M, R/\mathfrak{m}) \arrow[r] & \operatorname{Tor}_i^R(N, R/\mathfrak{m}) \arrow[r] & \operatorname{Tor}_i^R(R^{\oplus n}, R/\mathfrak{m}) \arrow[dll, out=-10, in=170] \\
& \operatorname{Tor}_i^R(M, R/\mathfrak{m}) \arrow[r] & \operatorname{Tor}_{i-1}^R(N, R/\mathfrak{m}) \arrow[r] & \operatorname{Tor}_{i-1}^R(R^{\oplus n}, R/\mathfrak{m}) \arrow[r] & \cdots
\end{tikzcd}$$

and which ends on the right with

$$\cdots \operatorname{Tor}_1^R(M, R/\mathfrak{m}) \to N \otimes_R R/\mathfrak{m} \to R^{\oplus n} \otimes_R R/\mathfrak{m} \to M \otimes_R R/\mathfrak{m} \to 0.$$

Since $\mathrm{Tor}_i^R(R^{\oplus n}, R/\mathfrak{m}) = 0$ for all $i \geq 1$, the long exact sequence above yields isomorphisms

$$\mathrm{Tor}_i(M, R/\mathfrak{m}) \cong \mathrm{Tor}_{i-1}(N, R/\mathfrak{m}) \tag{1.5.19}$$

for all $i \geq 2$. We're going to use this observation in a *dimension shifting* argument along with the characterization of projective dimension given by the equivalence of (2) and (3) from Proposition 1.5.16. For this we'll need the following lemma:

Lemma 1.5.20. *Let R be any ring, and let P and Q be two R-modules. Suppose that $x \in R$ has the following properties:*
(1) *x is not a zero divisor in R,*
(2) *the map $\cdot x : P \to P$ which sends $y \in P$ to yx has trivial kernel,*
(3) *$xQ = 0$.*
Then, for all $i \geq 0$, there is a canonical isomorphism

$$\mathrm{Tor}_i^R(P, Q) \cong \mathrm{Tor}_i^{R/(x)}(P/xP, Q).$$

Proof. By (1), the sequence

$$0 \to R \xrightarrow{\cdot x} R \to R/(x) \to 0 \tag{1.5.21}$$

is exact. Find a resolution of P by free R-modules $\{P_i\}_{i \geq 0}$ like

$$\cdots \to P_i \to \cdots \to P_0 \to P \to 0.$$

and let $\mathcal{P}_\bullet$ denote the complex omitting the rightmost term:

$$\cdots \to P_i \to \cdots \to P_0 \to 0.$$

Tensoring each of the R-modules P_i of the complex $\mathcal{P}_\bullet$ by the sequence (1.5.21) allows us to construct a short exact sequence of complexes

$$0 \to \mathcal{P}_\bullet \xrightarrow{\cdot x} \mathcal{P}_\bullet \to \mathcal{P}_\bullet \otimes_R R/(x) \to 0.$$

Accordingly, there is a long exact sequence in homology [Wei94, Theorem 1.3.1]

$$\cdots \to \mathrm{H}_n(\mathcal{P}_\bullet) \xrightarrow{\cdot x} \mathrm{H}_n(\mathcal{P}_\bullet) \to \mathrm{H}_n(\mathcal{P}_\bullet \otimes_R R/(x)) \to \mathrm{H}_{n-1}(\mathcal{P}_\bullet) \xrightarrow{\cdot x} \cdots \tag{1.5.22}$$

which ends with

$$\cdots \to H_1(\mathcal{P}_\bullet \otimes_R R/(x)) \to P \xrightarrow{\cdot x} P \to P \otimes_R R/(x) \to 0.$$

From (1.5.22) and our assumption (2), we find that $\mathcal{P}_\bullet \otimes_R R/(x)$ is then a free resolution of the $R/(x)$-module $P \otimes_R R/(x) \cong P/xP$. Now, by (3), we can consider Q as an $R/(x)$-module giving a canonical isomorphism of complexes

$$\mathcal{P}_\bullet \otimes_R R/(x) \otimes_{R/(x)} Q \cong \mathcal{P}_\bullet \otimes_R Q. \tag{1.5.23}$$

Taking homology of the complex on the left in (1.5.23) then gives

$$\mathrm{H}_i(\mathcal{P}_\bullet \otimes_R R/(x) \otimes_{R/(x)} Q) \cong \mathrm{Tor}_i^{R/(x)}(P/xP, Q)$$

while taking homology of the complex on the right gives

$$\mathrm{H}_i(\mathcal{P}_\bullet \otimes_R Q) \cong \mathrm{Tor}_i(P, Q)$$

producing a canonical isomorphism as claimed. $\square$

Coming back to the proof from before, we have that $(R, \mathfrak{m})$ is a regular local ring of Krull dimension $d > 0$. We let $x \in \mathfrak{m}$ be a part of a minimal generating set for $\mathfrak{m}$. Then $R/(x)$ is also a regular local ring and now $\mathrm{Kr.dim}(R/(x)) = d - 1$. Applying Lemma 1.5.20 with $P = N$ and $Q = R/\mathfrak{m}$ shows that

$$\mathrm{Tor}_{i-1}(N, R/\mathfrak{m}) \cong \mathrm{Tor}_{i-1}^{R/(x)}(N/xN, R/\mathfrak{m}).$$

Our induction hypothesis along with the equality of (2) and (3) from Proposition 1.5.16 imply that the latter vanishes if $i > d$ since $d \geq 1$. Therefore, because of the isomorphism in (1.5.19), we have that $\mathrm{Tor}_i(M, R/\mathfrak{m}) = 0$ if $i > d$. Since this implies that $\mathrm{pd}_R(M) \leq d$, by the equivalence between (2) and (3) of Proposition 1.5.16 again, we're done! $\square$

Corollary 1.5.24. *Suppose that $(R, \mathfrak{m})$ is a regular local ring with Krull dimension* $\mathrm{Kr.dim}(R) = d$. *Let M be a finitely generated R-module. Then* $\mathrm{pd}_R(M) \leq d$. $\square$

Corollary 1.5.25. *Suppose that $(R, \mathfrak{m})$ is a regular local ring with* $\mathrm{Kr.dim}(R) = d$. *Then $G(R) \cong \mathbb{Z}$ and* $\mathrm{Cl}(R) = 0$.

Proof. If M is a finitely generated R-module, then any minimal free resolution of M is finite by Corollary 1.5.24. If a minimal free resolution of M is given

$$0 \to R^{\oplus n_m} \to \cdots \to R^{\oplus n_0} \to M \to 0$$

then, by Remark 1.4.2, we there is an equality

$$[M] = \sum_{i=0}^{m} (-1)^i n_i [R]$$

inside $G(R)$. Since $(R, \mathfrak{m})$ is an integral domain, the rank map of Remark 1.4.9 then gives an isomorphism $G(R) \cong \mathbb{Z}$.

For the second claim we note, since R is integrally closed by Proposition 1.5.7, that it follows from Theorem 1.4.19 and its proof that $\mathrm{Cl}(R) = 0$. $\square$

Corollary 1.5.26. *If R is a regular ring, then R is locally factorial.*

Proof. For any prime ideal $\mathfrak{p} \subset R$, the local ring $(R_\mathfrak{p}, \mathfrak{p}R_\mathfrak{p})$ is integrally closed, by Proposition 1.5.7, and has $\mathrm{Cl}(R_\mathfrak{p}) = 0$, by Corollary 1.5.25. Hence $R_\mathfrak{p}$ is a UFD by Corollary 1.3.33. □

Example 1.5.27. Let k be an arbitrary field, and let $R = k[x, y, z]/(xy - z^2)$ be the coordinate ring of the cone from Example 1.3.37. Then R is not regular since the localization $R_\mathfrak{m}$ at the maximal ideal $\mathfrak{m} = (x, y, z)$ is not a regular local ring. Indeed, $\mathrm{Kr.\,dim}(R_\mathfrak{m}) = 2$ but we saw in Example 1.3.37 that $\dim_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) = 3$. This fact was used directly in our argument that R was not locally factorial.

The last implication of Theorem 1.5.12 is also proved by an induction argument. After we finish the proof of this implication, we will have completed the entire proof of Theorem 1.5.12. This will be the end of our study of the homological properties of regular local rings.

Proof of (3) $\implies$ (1). Suppose that $(R, \mathfrak{m})$ is a Noetherian local ring that has the property $\mathrm{pd}_R(R/\mathfrak{m}) < \infty$. We need to show that this implies that R is regular. As a first case, let's suppose that $\mathrm{Kr.\,dim}(R) = 0$ so that R is an Artinian local ring and $\mathfrak{m}$ is the unique prime ideal of R.

Then one of the following cases holds: either $\mathrm{pd}_R(R/\mathfrak{m}) = 0$ or $\mathrm{pd}_R(R/\mathfrak{m}) > 0$. If $\mathrm{pd}_R(R/\mathfrak{m}) = 0$, then $R/\mathfrak{m}$ is a free R-module which implies $\mathfrak{m} = 0$. Otherwise we have that $\mathrm{pd}_R(R/\mathfrak{m}) > 0$ and any minimal free resolution for $R/\mathfrak{m}$ ends on the left with an injection

$$0 \to R^{\oplus n} \xrightarrow{\phi} R^{\oplus m} \to \cdots$$

having the property that $\phi \otimes \mathrm{id}_{R/\mathfrak{m}} = 0$ and for some $m, n \geq 1$. However, if $\mathfrak{m} \neq 0$, then any nonzero element $0 \neq x \in \mathfrak{m}$ is nilpotent and it follows that no such injection can exist. Indeed, if we represent a homomorphism $\phi : R^{\oplus n} \to R^{\oplus m}$ as an $m \times n$-matrix with respect to the standard bases for $R^{\oplus n}$ and $R^{\oplus m}$ then

$$\phi = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{m1} & \cdots & x_{mn} \end{pmatrix}$$

and, if $\phi \otimes \mathrm{Id}_{R/\mathfrak{m}} = 0$, we must have that $x_{ij} \in \mathfrak{m}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$ by the comments above Remark 1.5.15. For suitable integers $r_1, ..., r_n > 0$, the column vector $v = (x_{11}^{r_1} \cdots x_{1n}^{r_n})^T$ is then in the kernel of ϕ.

Now assume that, for every local Noetherian ring $(R, \mathfrak{m})$ with $\mathrm{Kr.\,dim}(R) < d$, the assumption $\mathrm{pd}_R(R/\mathfrak{m}) < \infty$ implies that R is regular. For our induction step, we suppose that $(R, \mathfrak{m})$ is an arbitrary Noetherian local ring with Krull dimension

$1 \leq \mathrm{Kr.dim}(R) = d$ having the property that $\mathrm{pd}_R(R/\mathfrak{m}) < \infty$. We want to prove that such a ring R is necessarily regular.

In order to use our induction hypothesis, we first show that there exists an element $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ which is not a zero divisor in R. The set of all zero divisors in R is precisely the union of those prime ideals $\mathfrak{p}_i$ which occur as radicals of the primary ideals $\mathfrak{q}_i$ appearing in a primary decomposition of the zero ideal $(0) = \bigcap_{i=1}^s \mathfrak{q}_i$ by [AM69, Proposition 4.7]. So, if there was a containment

$$\mathfrak{m} \subset \bigcup_{i=1}^{s} \mathfrak{p}_i \cup \mathfrak{m}^2,$$

then, since $\mathfrak{m}$ is not contained in $\mathfrak{m}^2$ as $\mathrm{Kr.dim}(R) \geq 1$ [AM69, Proposition 8.6], we must have $\mathfrak{m} = \mathfrak{p}_i$ for some $1 \leq i \leq s$ by prime avoidance [Sta19, Tag 00DS]. However, if every element of $\mathfrak{m}$ is a zero divisor then, similar to the proof of the zero dimensional case, no minimal free resolution of $R/\mathfrak{m}$ can be finite as any homomorphism $\phi : R^{\oplus n} \to R^{\oplus m}$ with $\phi \otimes \mathrm{id}_{R/\mathfrak{m}} = 0$ has nontrivial kernel.

If $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ is any element which is not a zero divisor of R, then $R/(x)$ is a local Noetherian ring and $\mathrm{Kr.dim}(R/(x)) = d - 1$ by [AM69, Corollary 11.18]. The maximal ideal of $R/(x)$ is $\mathfrak{m}_1 = \mathfrak{m}/(x)$ and the residue field of $R/(x)$ is $R/\mathfrak{m}$. So, if we can show that $\mathrm{pd}_{R/(x)}(R/\mathfrak{m}) < \infty$, then the induction hypothesis implies that $R/(x)$ is regular. Since we can see directly that

$$\dim_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) = \dim_{R/\mathfrak{m}}(\mathfrak{m}_1/\mathfrak{m}_1^2) + 1$$

it follows if $R/(x)$ is regular, then R is regular as well. So, to complete the proof, it suffices to show that $\mathrm{pd}_{R/(x)}(R/\mathfrak{m}) < \infty$. This is a direct consequence of a more general claim which is proved in the next lemma (set $M = R/\mathfrak{m}$). □

Lemma 1.5.28. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, and let M be a finitely generated R-module. Suppose that an element $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ satisfies the following properties:*
(1) *x is not a zero divisor in R,*
(2) *$xM = 0$.*
Then, if M has finite projective dimension as an R-module, we also have that M has finite projective dimension as an $R/(x)$-module.

Proof. Since $xM = 0$, we must have $\mathrm{pd}_R(M) \geq 1$. We're going to reduce to the case that $\mathrm{pd}_R(M) = 1$. So, suppose that $\mathrm{pd}_R(M) > 1$. Then, considering M as a finitely generated $R/(x)$-module, we can find a short exact sequence

$$0 \to K \to (R/(x))^{\oplus n} \to M \to 0 \tag{1.5.29}$$

with n a positive integer and K the appropriate kernel. As x is not a zero divisor in R, the R-module $(R/(x))^{\oplus n}$ has projective dimension equal to 1 with an explicit resolution given by a direct sum of n copies of the sequence

$$0 \to R \xrightarrow{\cdot x} R \to R/(x) \to 0.$$

Considering K as an R-module, this gives an equality $\mathrm{pd}_R(K) = \mathrm{pd}_R(M) - 1$ by Corollary 1.5.17. But, if $\mathrm{pd}_{R/(x)}(K)$ is finite then $\mathrm{pd}_{R/(x)}(M)$ is also finite because we can concatenate an $R/(x)$-resolution of K with the sequence (1.5.29). Hence, up to replacing M by K and repeating this process, we can assume $\mathrm{pd}_R(M) = 1$.

Once we're in the case that $\mathrm{pd}_R(M) = 1$, the proof is entirely linear algebra. Choose a minimal free resolution for M like

$$0 \to R^{\oplus n} \xrightarrow{\phi} R^{\oplus m} \xrightarrow{j} M \to 0 \tag{1.5.30}$$

with some integers $m, n \geq 1$. Localizing at x yields an exact sequence

$$0 \to R_x^{\oplus n} \xrightarrow{\phi \otimes \mathrm{Id}_{R_x}} R_x^{\oplus m} \to 0$$

since $M \otimes_R R_x = 0$. From this we find that $n = m$.

For any integer i with $1 \leq i \leq n$, if we denote by e_i the ith standard basis element for $R^{\oplus n}$, then

$$j(xe_i) = xj(e_i) = 0$$

since $xM = 0$. By exactness of (1.5.30), this means that there is a $v_i \in R^{\oplus n}$ with $\phi(v_i) = xe_i$. Let ψ be the $n \times n$-matrix whose ith column is the vector v_i so that, if we represent ϕ with an $n \times n$-matrix with coefficients in $\mathfrak{m}$, we have $\phi\psi = xI_n$ for the $n \times n$-identity matrix I_n.

From the equality $\phi\psi = xI_n$, it follows that ψ is invertible; proof of this claim, which requires only some elementary linear algebra, is outlined in Exercise 1.5.2. Altogether, this implies that there is a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & R^{\oplus n} \arrow[r, "\cdot x"] \arrow[d, "\psi"] & R^{\oplus n} \arrow[r] \arrow[d, equal] & (R/(x))^{\oplus n} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R^{\oplus n} \arrow[r, "\phi"] & R^{\oplus n} \arrow[r, "j"] & M \arrow[r] & 0.
\end{tikzcd}$$

Since the vertical arrows on the left and in the middle are both isomorphisms, it follows that the vertical arrow on the right is an isomorphism too. So M had finite projective dimension as an $R/(x)$-module after all. □

It's useful, in practice, to have a way to check whether a given ring R is regular without needing to check that equality in (1.5.1) holds for the localization $(R_\mathfrak{p}, \mathfrak{p}R_\mathfrak{p})$ at each prime ideal $\mathfrak{p} \subset R$. Together, the next corollary and the following remark give an often-times more efficient method for checking whether the coordinate ring of a variety over an algebraically closed field is regular.

Corollary 1.5.31. *If $(R, \mathfrak{m})$ is a regular local ring and, if $\mathfrak{p} \subset R$ is a prime ideal, then $R_{\mathfrak{p}}$ is also a regular local ring. Thus, the following are equivalent conditions for a ring S:*

(1) *S is regular,*
(2) *$S_{\mathfrak{p}}$ is a regular local ring for all prime ideals $\mathfrak{p} \subset S$,*
(3) *$S_{\mathfrak{m}}$ is a regular local ring for all maximal ideals $\mathfrak{m} \subset S$.*

Proof. Let $\mathfrak{p} \subset R$ be a prime ideal. Since R is regular, any minimal free resolution of the R-module $R/\mathfrak{p}$ is finite:

$$0 \to R^{\oplus n_m} \to \cdots \to R^{\oplus n_0} \to R/\mathfrak{p} \to 0.$$

If we localize such a resolution at the prime $\mathfrak{p}$, then we get a finite resolution of the $R_{\mathfrak{p}}$-module $(R/\mathfrak{p}) \otimes_R R_{\mathfrak{p}} \cong R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. As this is the residue field of the local ring $R_{\mathfrak{p}}$, it follows that $\mathrm{pd}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}) < \infty$. Hence $R_{\mathfrak{p}}$ is regular by Theorem 1.5.12. □

Remark 1.5.32. Let k be an algebraically closed field and let $R = k[x_1, ..., x_n]/I$ be the quotient of the given polynomial ring over k by an ideal $I = (f_1, ..., f_m)$. By Hilbert's Nullstellensatz [Rei95, §5.3, Proposition], an ideal $\mathfrak{m}$ of R is maximal if and only if there are elements $a_1, ..., a_n \in k$ so that both

$$\mathfrak{m} = (x_1 - a_1, ..., x_n - a_n)$$

and there is simultaneous vanishing

$$f_1(a_1, ..., a_n) = 0, \ \ldots, \ f_m(a_1, ..., a_n) = 0.$$

If R is as above and $\mathfrak{m} = (x_1 - a_1, ..., x_n - a_n)$ is a specific maximal ideal, then one can show that the local ring $R_{\mathfrak{m}}$ is regular if and only if the Jacobian matrix

$$J_{\mathbf{a}} = \left(\frac{\partial f_i}{\partial x_j}(\mathbf{a})\right)_{1 \le i \le m, 1 \le j \le n} \qquad \text{where } \mathbf{a} = (a_1, ..., a_n)$$

has rank $J_{\mathbf{a}} = n - \mathrm{ht}(\mathfrak{m})$, see [Liu02, Ch. 4, §2, Theorem 2.19]. See Exercise 1.5.3 for an extension of this result along with an example.

Global properties of regular rings

With the local theory of regular rings taken care of, we are now ready to globalize our results.

Proposition 1.5.33. *Let R be a Noetherian ring, and let M be a finitely generated R-module. Then the following are equivalent:*

(1) *there is an upper bound $\mathrm{pd}_R(M) \le n$ on the projective dimension of M,*

(2) *for every R-module N, we have* $\mathrm{Ext}_R^{n+1}(M,N)=0$,
(3) *the functor* $\mathrm{Ext}_R^n(M,-)$ *is right exact,*
(4) *if whenever there is an exact sequence of R-modules*

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

such that P_i is projective for all $i<n$, then P_n is projective.

Proof. Assume (1), so that there is a finite resolution

$$0 \to P_n \to \cdots \to P_0 \to M \to 0$$

with some finitely generated and projective R-modules P_i for $0 \le i \le n$. If N is an arbitrary R-module, then $\mathrm{Ext}_R^i(M,N)$ can be computed by omitting M from the above resolution, applying the functor $\mathrm{Hom}_R(-,N)$, and taking homology at the ith position of the resulting complex. Hence $\mathrm{Ext}_R^{n+1}(M,N)=0$, showing (2).

Assume (2), and let

$$0 \to A \to B \to C \to 0$$

be a short exact sequence. Then the long exact sequence of $\mathrm{Ext}_R^*(M,-)$ associated to this short exact sequence ends with

$$\cdots \to \mathrm{Ext}_R^n(M,A) \to \mathrm{Ext}_R^n(M,B) \to \mathrm{Ext}_R^n(M,C) \to \mathrm{Ext}_R^{n+1}(M,A)=0.$$

Thus (3) holds.

Now assume (3). If $n=0$, then (4) immediately follows from Remark 1.1.2. So, assume $n>0$ and let

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0 \tag{1.5.34}$$

be an exact sequence with P_i projective for all $0 \le i < n$. We want to show that P_n is projective. We do this by checking directly that, given a surjection $L \to N$, the R-module P_n satisfies the necessary lifting condition of Definition 1.1.1.

Truncating the sequence (1.5.34) on the left, we get exact sequences

$$0 \to P_n \to P_{n-1} \to P'_{n-1} \to 0 \quad \text{and} \quad 0 \to P'_{n-1} \to P_{n-2} \to \cdots \to M \to 0.$$

The sequence on the left above provides us with a connecting homomorphism

$$\mathrm{Hom}_R(P_n,L) \to \mathrm{Ext}_R^1(P'_{n-1},L)$$

which is surjective, since $\mathrm{Ext}_R^1(P_{n-1},L)=0$ as P_{n-1} is projective by assumption, and which has kernel exactly the image of $\mathrm{Hom}_R(P_{n-1},L)$ in $\mathrm{Hom}_R(P_n,L)$. If we truncate the sequence on the right in the above, we obtain a short exact sequence

$$0 \to P'_{n-1} \to P_{n-2} \to P'_{n-2} \to 0$$

from which we get a connecting homomorphism

$$\mathrm{Ext}^1_R(P'_{n-1}, L) \to \mathrm{Ext}^2_R(P'_{n-2}, L).$$

Moreover, since P_{n-2} is projective by assumption, this latter connecting morphism is an isomorphism. Continuing in this fashion and composing all of the maps gives an exact sequence

$$\mathrm{Hom}_R(P_{n-1}, L) \to \mathrm{Hom}_R(P_n, L) \to \mathrm{Ext}^n(M, L) \to 0.$$

Doing the same procedure with N instead of L yields a similar short exact sequence and, by the functorality of the connecting homomorphism, these two sequences fit into a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_R(P_{n-1}, L) \arrow[r] \arrow[d] & \mathrm{Hom}_R(P_n, L) \arrow[r] \arrow[d] & \mathrm{Ext}^n(M, L) \arrow[r] \arrow[d] & 0 \\
\mathrm{Hom}_R(P_{n-1}, N) \arrow[r] & \mathrm{Hom}_R(P_n, N) \arrow[r] & \mathrm{Ext}^n(M, N) \arrow[r] & 0.
\end{tikzcd}$$

By assumption, the rightmost vertical arrow in this diagram is a surjection. Since P_{n-1} is assumed projective, the leftmost vertical arrow is also a surjection. By a diagram chase, it follows that the middle vertical arrow is then a surjection. Hence P_n is projective as claimed, proving (4).

Finally, assume (4). Since M is finitely generated, we can find an integer $j_0 > 0$ and a surjection $R^{\oplus j_0} \to M$. Since R is Noetherian, the kernel of this map is also finitely generated. Hence we can construct an exact sequence

$$0 \to K \to R^{\oplus j_{n-1}} \to \cdots \to R^{\oplus j_0} \to M \to 0$$

with K an appropriate kernel. It follows from (4) that K is a projective R-module. Since R is Noetherian, we must also have that K is a finitely generated R-module. Hence $\mathrm{pd}_R(M) \leq n$, which is (1). $\square$

The next result shows the truly extraordinary connection between homological algebra and algebraic geometry. It follows directly from, and it can be seen as a partial globalization of, the Auslander-Buchsbaum theorem (Theorem 1.5.12).

Theorem 1.5.35. *Suppose that R is a Noetherian regular ring of Krull dimension $\mathrm{Kr.dim}(R) = d$ for some integer $d \geq 0$. Let M be a finitely generated R-module. Then $\mathrm{pd}_R(M) \leq d$.*

Proof. In order to prove the theorem it's sufficient, by Proposition 1.5.33, to check that $\mathrm{Ext}^{d+1}_R(M, N) = 0$ for every R-module N. So let N be an arbitrary R-module and let

$$\cdots \to P_i \to \cdots \to P_0 \to M \to 0 \tag{1.5.36}$$

be a resolution of M by finitely generated and projective R-modules P_i (such a resolution exists since R is Noetherian and M is finitely generated).

Now we can go in two different directions. We can either first localize (1.5.36) at a prime ideal $\mathfrak{p} \subset R$, and then apply the functor $\mathrm{Hom}_{R_\mathfrak{p}}(-, N_\mathfrak{p})$. Or, alternatively, we can apply the functor $\mathrm{Hom}_R(-, N)$ to (1.5.36) and then localize at $\mathfrak{p} \subset R$. Both routes produce a complex of $R_\mathfrak{p}$-modules and the two complexes are comparable by the canonical isomorphisms of Lemma 1.1.9, i.e. there is a commutative ladder with complexes for rows like:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{Hom}_R(P_{i+1}, N)_\mathfrak{p} \arrow[r] \arrow[d, equal] & \mathrm{Hom}_R(P_i, N)_\mathfrak{p} \arrow[r] \arrow[d, equal] & \mathrm{Hom}_R(P_{i-1}, N)_\mathfrak{p} \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & \mathrm{Hom}_{R_\mathfrak{p}}((P_{i+1})_\mathfrak{p}, N_\mathfrak{p}) \arrow[r] & \mathrm{Hom}_{R_\mathfrak{p}}((P_i)_\mathfrak{p}, N_\mathfrak{p}) \arrow[r] & \mathrm{Hom}_{R_\mathfrak{p}}((P_{i-1})_\mathfrak{p}, N_\mathfrak{p}) \arrow[r] & \cdots .
\end{tikzcd}$$

This implies, in particular, that for each $i \geq 1$ there is a canonical isomorphism

$$\mathrm{Ext}^i_R(M, N) \otimes_R R_\mathfrak{p} \xrightarrow{\sim} \mathrm{Ext}^i_{R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}). \tag{1.5.37}$$

Since R is regular with $\mathrm{Kr.dim}(R) = d$, the local ring $R_\mathfrak{p}$ satisfies the conditions of Corollary 1.5.24. From Proposition 1.5.33 applied to $R_\mathfrak{p}$, this gives the vanishing

$$\mathrm{Ext}^{d+1}_{R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) = 0 \qquad \text{for all prime ideals } \mathfrak{p} \subset R.$$

Together with (1.5.37), this implies $\mathrm{Ext}^{d+1}_R(M, N) = 0$ as desired. □

We now have all of the homological tools needed to prove the main theorem of this section: Theorem 1.5.3. Recall that there is a canonical group homomorphism

$$\varphi_R : K(R) \to G(R) \qquad [P] \mapsto [P]$$

induced by the inclusion $P_{fg}(R) \subset M_{fg}(R)$ of the free abelian group $P_{fg}(R)$ on isomorphism classes of finitely generated projective R-modules into the free abelian group $M_{fg}(R)$ on isomorphism classes of all finitely generated R-modules.

Proof of Theorem 1.5.3. As per the statement of the theorem, we're assuming that R is a regular ring of finite Krull dimension and we want to show that the canonical homomorphism φ_R is an isomorphism. To do this, we construct an inverse to φ_R. For any finitely generated R-module M, choose a finite resolution of M by finitely generated and projective R-modules (which exists by Theorem 1.5.35):

$$0 \to P_n \to \cdots \to P_0 \to M \to 0. \tag{1.5.38}$$

We can then define a group homomorphism

$$\overline{\rho} : M_{fg}(R) \to K(R) \qquad M \mapsto \sum_{i \geq 0} (-1)^i [P_i].$$

If we can show that $\overline{\rho}$ descends to map $\rho : G(R) \to K(R)$, i.e. if we can show that there is an equality $\overline{\rho}(M) = \overline{\rho}(L) + \overline{\rho}(N)$ for any short exact sequence of finitely generated R-modules

$$0 \to L \to M \to N \to 0,$$

then ρ will clearly be the desired inverse. The proof now has two main-steps which, in both cases, reduce to a formal argument in homological algebra.

First, we observe that if we are given any other finite resolution of a fixed finitely generated R-module M, say

$$0 \to P'_m \to \cdots \to P'_0 \to M \to 0,$$

then this resolution defines the same element of $K(R)$ as the one used in (1.5.38) to define $\overline{\rho}(M)$, i.e. there is an equality

$$\overline{\rho}(M) = \sum_{i \geq 0} (-1)^i [P_i] = \sum_{i \geq 0} (-1)^i [P'_i] \tag{1.5.39}$$

as elements of $K(R)$. To see this, denote by $\mathcal{P}_\bullet$ and $\mathcal{P}'_\bullet$ the two resolutions (omitting the M term). Then, since both $\mathcal{P}_\bullet$ and $\mathcal{P}'_\bullet$ are made up of projective R-modules, there is a quasi-isomorphism of complexes $f : \mathcal{P}_\bullet \to \mathcal{P}'_\bullet$ by [Wei94, Theorem 2.2.6]. The mapping cone $\mathcal{C}(f)_\bullet$ of f is then an exact complex which looks like

$$\cdots \to P_i \oplus P'_{i+1} \to P_{i-1} \oplus P'_i \to P_{i-2} \oplus P'_{i-1} \to \cdots \to P'_0 \to 0.$$

Hence, applying Exercise 1.2.4 to $\mathcal{C}(f)_\bullet$, we get an equality

$$0 = [P'_0] + \sum_{i \geq 1} (-1)^i [P_{i-1} \oplus P'_i] = \sum_{i \geq 0} (-1)^i [P'_i] - \sum_{i \geq 0} (-1)^i [P_i]$$

which immediately implies the equality in (1.5.39).

Now suppose that we're given a short exact sequence of finitely generated R-modules such as

$$0 \to L \to M \to N \to 0.$$

Assume that the resolution of L used to define $\overline{\rho}(L)$ is

$$0 \to \cdots \to P'_i \to \cdots \to P'_0 \to L \to 0,$$

with finitely generated projective R-modules P'_i, and similarly for N

$$0 \to \cdots \to P''_j \to \cdots \to P''_0 \to N \to 0$$

with finitely generated projective R-modules P''_j. The Horseshoe lemma [Wei94, Lemma 2.2.8] then shows that there is a finite resolution of M with terms the sum of those terms from the given resolutions of L and N,

$$0 \to \cdots \to P'_i \oplus P''_i \to \cdots \to P'_0 \oplus P''_0 \to M \to 0. \tag{1.5.40}$$

Since the class of $\overline{\rho}(M)$ inside $K(R)$ is independent of the choice of finite resolution, by the equality from (1.5.39), it follows from the existence of the resolution in (1.5.40) that $\overline{\rho}(L) + \overline{\rho}(N) = \overline{\rho}(M)$ as we had indicated. $\square$

Dedekind domains

We conclude, here in the last part to this section, with the following result on the relationship between K-theory and G-theory of Dedekind domains.

Theorem 1.5.41. *Let R be a fixed Dedekind domain. Then there is a commutative diagram with exact rows*

$$\begin{tikzcd} 1 \arrow[r] & \mathrm{Pic}(R) \arrow[r] \arrow[d, "c_1"] & K(R) \arrow[r, "\mathrm{rk}"] \arrow[d, "\varphi_R"] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathrm{Cl}(R) \arrow[r] & G(R) \arrow[r, "\mathrm{rk}"] & \mathbb{Z} \arrow[r] & 0. \end{tikzcd} \tag{1.5.42}$$

Here the leftmost vertical arrow $c_1 : \mathrm{Pic}(R) \to \mathrm{Cl}(R)$ is the homomorphism of Theorem 1.3.27 and the middle vertical arrow $\varphi_R : K(R) \to G(R)$ is the canonical homomorphism satisfying $\varphi_R([P]) = [P]$.

Theorem 1.5.41 doesn't rely on the fact that φ_R is an isomorphism for a Dedekind domain R (which is a consequence of Theorem 1.5.3 and Example 1.5.6). Since we know that the homomorphism $c_1 : \mathrm{Pic}(R) \to \mathrm{Cl}(R)$ is an isomorphism for a Dedekind domain R (because R is locally factorial, Remark 1.3.32), this theorem gives an alternative proof that φ_R is an isomorphism for any Dedekind domain R.

There are still two maps in the diagram (1.5.42) whose definition deserves explanation. First, the existence of an injective homomorphism $\mathrm{Cl}(R) \to G(R)$ is a direct consequence of Theorem 1.4.19 and its proof; specifically, in this case we have $F^2_\tau G(R) = 0$ since $\mathrm{Kr.dim}(R) = 1$, as R is a Dedekind domain, so that $\mathrm{Cl}(R)$ is isomorphic with the subgroup of $G(R)$ generated by classes $[R/\mathfrak{p}]$ for nonzero prime ideals $\mathfrak{p} \subset R$. As for the remaining map $\mathrm{Pic}(R) \to K(R)$, let's try to guess how this arrow should be defined (rather than giving the definition outright).

Since we expect that we should be able to show that φ_R is an isomorphism, if we take an element $[R/\mathfrak{p}] \in G(R)$ for a nonzero prime ideal $\mathfrak{p} \subset R$ then there should be an element of $K(R)$ mapping to $[R/\mathfrak{p}]$. A finite resolution of $R/\mathfrak{p}$ by projective R-modules is easy to find since every nonzero ideal in a Dedekind domain is finitely generated and projective:

$$0 \to \mathfrak{p} \to R \to R/\mathfrak{p} \to 0.$$

As we also have $c_1([\mathfrak{p}]) = [R/\mathfrak{p}]$, we must have that the map $\mathrm{Pic}(R) \to K(R)$ is defined by sending the class of a prime ideal $[\mathfrak{p}]$ to $[R] - [\mathfrak{p}]$. To check that this is a well-defined homomorphism takes some work (and a couple lemmas).

Lemma 1.5.43. *Let R be any Dedekind domain and let M be a finitely generated projective R-module. Then there exist ideals $J_1, ..., J_n \subset R$ and an isomorphism of R-modules*

$$M \cong J_1 \oplus \cdots \oplus J_n.$$

In particular, any locally free R-module M of finite rank is isomorphic to a direct sum of invertible modules.

Proof. The proof is by induction. By Theorem 1.1.10 we know that M is a direct summand of a free module $R^{\oplus n}$ for some $n \geq 1$. The basis for our induction is the case when $n = 1$. In this case, we find that M is an R-submodule of R itself, so M is isomorphic with an ideal of R.

Assume that, for all $1 \leq k < n$, every finitely generated projective R-module with an embedding into $R^{\oplus k}$ is isomorphic to a sum $M \cong J_1 \oplus \cdots \oplus J_k$ of ideals $J_1, ..., J_k \subset R$. If $M \neq 0$ is now embedded into $R^{\oplus n}$, then we can project to the last coordinate to get a map

$$\pi : M \subset R^{\oplus n} \to R.$$

If $\pi(M) = 0$, then M actually embeds into the first summand of $R^{\oplus n-1} \oplus R = R^{\oplus n}$ and we can conclude by the induction hypothesis. If $\pi(M) \neq 0$ then $\pi(M) = J_n$ is a nonzero ideal of R which is also finitely generated and projective by Lemma 1.3.13 and Proposition 1.3.4 since R is a Dedekind domain.

By the definition of projectivity, we get a map $J_n \to M$ fitting into the following commutative diagram below.

$$\begin{tikzcd}
 & J_n \arrow[dl, dotted] \arrow[d, equal] & \\
M \arrow[r, "\pi"] & J_n \arrow[r] & 0
\end{tikzcd}$$

The splitting lemma shows that M is a direct sum $M \cong K \oplus J_n$ with $K = \ker(\pi)$. As a summand of M, the R-module K is also finitely generated and projective. The inclusion $M \subset R^{\oplus n}$ embeds K into the first summand of $R^{\oplus n-1} \oplus R = R^{\oplus n}$. So by the induction hypothesis, there is an isomorphism $K \cong J_1 \oplus \cdots \oplus J_{n-1}$ for some ideals $J_1, ..., J_{n-1} \subset R$ and thus $M \cong J_1 \oplus \cdots \oplus J_n$. $\square$

Lemma 1.5.44. *Let R be a Dedekind domain with field of fractions F. Then for any pair of fractional ideals $I, J \subset F$ there is an isomorphism of R-modules*

$$I \oplus J \cong R \oplus IJ.$$

Hence also for any $n \geq 2$ fractional ideals $I_1, ..., I_n \subset F$ there is an isomorphism

$$I_1 \oplus \cdots \oplus I_n \cong R^{n-1} \oplus I_1 \cdots I_n.$$

Proof. We first prove the unrelated claim that if I, J are two fractional ideals for a Dedekind domain R, then there are elements $x, y \in F$ so that $xI, yJ \subset R$ are two relatively prime ideals, i.e. xI and yJ are both ideals of R and $xI + yJ = R$.

To do this, we can assume that both I and J are already ideals of R, and not just fractional ideals for R, by multiplying I and J by suitable elements of R. In this case, let $J = \mathfrak{p}_1^{r_1} \cdots \mathfrak{p}_s^{r_s}$ be the unique prime decomposition of the ideal J with $r_i > 0$ for each $1 \leq i \leq s$. For each such i, pick an element a_i so that

$$a_i \in I^{-1}\mathfrak{p}_1 \cdots \mathfrak{p}_{i-1}\mathfrak{p}_{i+1} \cdots \mathfrak{p}_s \setminus I^{-1}\mathfrak{p}_1 \cdots \mathfrak{p}_s.$$

Let $I' = a_i R$ be the fractional ideal generated by a_i. We have that

$$a_i I = II' \subset II^{-1}\mathfrak{p}_1 \cdots \mathfrak{p}_{i-1}\mathfrak{p}_{i+1} \cdots \mathfrak{p}_s = \mathfrak{p}_1 \cdots \mathfrak{p}_{i-1}\mathfrak{p}_{i+1} \cdots \mathfrak{p}_s,$$

and

$$a_i I = II' \not\subset II^{-1}\mathfrak{p}_1 \cdots \mathfrak{p}_s = \mathfrak{p}_1 \cdots \mathfrak{p}_s$$

hence $a_i I \subset \mathfrak{p}_j$ for all $j \neq i$. Moreover, we have $a_i I \not\subset \mathfrak{p}_i$ since, if $a_i I \subset \mathfrak{p}_i$, then

$$a_i I \subset \bigcap_{i=1}^{s} \mathfrak{p}_i = \mathfrak{p}_1 \cdots \mathfrak{p}_s$$

with equality on the right from [AM69, Proposition 1.10].

If $s = 1$ then we're done. Otherwise, let $a = \sum_{i=1}^{s} a_i$. Then aI is an ideal since

$$aI \subset \sum_{i=1}^{s} a_i I = R.$$

If there was a containment $aI \subset \mathfrak{p}_i$ for some i, then we'd find that

$$a_i I \subset aI + \sum_{j \neq i} a_j I \subset \mathfrak{p}_i,$$

so $aI \not\subset \mathfrak{p}_i$ for any $1 \leq i \leq s$. Thus aI and J are relatively prime.

To prove the first claim of the lemma, let I, J be fractional ideals for R. By the previous paragraph, we can find $x, y \in F$ so that $I \cong xI = P$ and $J \cong yJ = Q$ are relatively prime ideals in R. Then $IJ \cong PQ$ and, by applying [AM69, Proposition 1.10] again, we have equality $PQ = P \cap Q$ so that there is an exact sequence of R-modules

$$0 \to PQ \xrightarrow{x \mapsto (x,x)} P \oplus Q \xrightarrow{(x,y) \mapsto x-y} R \to 0$$

which is, moreover, right-split since R is free. The second claim of the lemma follows immediately from the first, so we're done. $\square$

Proof of Theorem 1.5.41. We're going to prove that the map

$$h : \mathbb{Z} \oplus \mathrm{Pic}(R) \to K(R),$$

defined by sending a pair $(n, [L])$ of the sum $\mathbb{Z}\oplus\mathrm{Pic}(R)$ to $n[R]+[R]-[L]$ in $K(R)$, is an isomorphism. The top row of the diagram in (1.5.42) is then given by the inclusion $\mathrm{Pic}(R) \subset K(R)$, via this map h, followed by the projection from $K(R)$ to $\mathbb{Z}$. Checking commutativity of the diagram was done above Lemma 1.5.43.

First of all, the map h is a group homomorphism: if we have two arbitrary elements $(n_1, [L_1])$ and $(n_2, [L_2])$ of $\mathbb{Z} \oplus \mathrm{Pic}(R)$, then

$$\begin{aligned} h((n_1, [L_1])) + h((n_2, [L_2])) &= n_1[R] + [R] - [L_1] + n_2[R] + [R] - [L_2] \\ &= (n_1 + n_2)[R] + 2[R] - [L_1 \oplus L_2] \\ &= (n_1 + n_2)[R] + [R] - [L_1 \otimes L_2] \\ &= h((n_1 + n_2, [L_1 \otimes L_2])). \end{aligned}$$

Here we've used Lemmas 1.5.44 and 1.3.6 to substitute the relation

$$[L_1 \oplus L_2] = [R] + [L_1 \otimes L_2]$$

when going from the second to third equality (noting that $L_1 \otimes L_2 \cong L_1L_2$ since every ideal of R is flat as an R-module).

Now the map h is surjective: if M is any locally free R-module of finite rank, then by Lemma 1.5.43 there are ideals $I_1, ..., I_n$ of R, and with $n = \mathrm{rk}(M)$, together with an isomorphism $M \cong I_1 \oplus \cdots \oplus I_n$. Hence

$$\begin{aligned} h((-n, [I_1 \otimes \cdots \otimes I_n])) &= (1 - n)[R] - [I_1 \otimes \cdots \otimes I_n] \\ &= (1 - n)[R] + (n - 1)[R] - [I_1 \oplus \cdots \oplus I_n] = -[M]. \end{aligned}$$

The map h is also injective. If $x = (n, L)$ is in the kernel of h then

$$0 = \mathrm{rk}(0) = \mathrm{rk}(h(x)) = \mathrm{rk}(n[R] + [R] - [L^{-1}]) = n.$$

Comparing with the determinant homomorphism $\det : K(R) \to \mathrm{Pic}(R)$ gives

$$[R] = \det(0) = \det(h((0, [L]))) = \det([R] - [L]) = [L^{-1}]$$

as elements of $\mathrm{Pic}(R)$, hence $R \cong L$ as R-modules. □

EXERCISES FOR SECTION 1.5

1. Let k be a field. Find a k-algebra presentation for the coordinate ring of the tangent cone of the nodal cubic, with coordinate ring $R = k[x, y]/(y^2 - x^3 - x^2)$, at the origin. Draw a picture of the nodal cubic and its tangent cone.

2. Let $(R, \mathfrak{m})$ be an arbitrary local ring. Suppose there is an element $x \in \mathfrak{m} \setminus \mathfrak{m}^2$. Let ϕ be an $n \times n$-matrix with coefficients in $\mathfrak{m}$ for some $n \geq 1$, let ψ be an $n \times n$-matrix with coefficients in R, and suppose that there is an equality

$$\phi\psi = xI_n$$

with I_n the $n \times n$-identity matrix. We're going to show, in this exercise, that given assumptions imply that ψ is invertible. This result is used in the proof of Lemma 1.5.28 (and therefore also in the proof of Theorem 1.5.12).

(a) The proof of this claim goes by induction on n with the case $n = 1$ being used to start the induction. Prove the claim in the case $n = 1$.

(b) Assume now that the claim holds for all square matrices up to size $n \times n$. We need to reduce from the $n \times n$-matrix case to the case of matrices of size $(n-1) \times (n-1)$. To do this, note that ψ does not have all of its coefficients contained in $\mathfrak{m}$. Hence, up to multiplication by $n \times n$-invertible matrices P and Q, we can assume that there is an equality

$$\psi' = P\psi Q = \begin{pmatrix} 1 & 0 \\ 0 & \psi_0 \end{pmatrix}$$

with ψ_0 a matrix of size $(n-1) \times (n-1)$ with coefficients in R.

(c) Set $\phi' = Q^{-1}\phi P^{-1}$ and note ϕ' has coefficients in $\mathfrak{m}$. There is an equality

$$\phi'\psi' = (Q^{-1}\phi P^{-1})(P\psi Q) = xI_n.$$

Simultaneously, if we write

$$\phi' = \begin{pmatrix} m_{11} & v^T \\ u & \phi_0 \end{pmatrix},$$

where m_{11} is an element of $\mathfrak{m}$, where both v and u are $(n-1) \times 1$-column vectors, and with ϕ_0 a $(n-1) \times (n-1)$-matrix, then

$$\phi'\psi' = \begin{pmatrix} m_{11} & v^T \\ u & \phi_0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \psi_0 \end{pmatrix} = \begin{pmatrix} m_{11} & 0 \\ 0 & \phi_0\psi_0 \end{pmatrix}.$$

Therefore, we must have $\phi_0\psi_0 = xI_{n-1}$. Conclude that ψ is invertible.

3. (a) Let R and S be two Noetherian rings, and suppose that there is a ring map $f : R \to S$ which gives S the structure of a faithfully flat R-module. Suppose that S is regular. Prove that R is then regular as well. (Hint: reduce to the case where R and S both local rings).

(b) Let k be an arbitrary field with a fixed choice of algebraic closure k^a, and let R be a fixed finitely generated k-algebra. By part (a) of this exercise, if $R \otimes_k k^a$ is regular, then R is regular too. Use this observation, along with Remark 1.5.32, to show that the ring $k[x, y, z]/(xy - z^2 + 1)$ is regular for any field k of characteristic not 2. Compare with Example 1.2.16.

(c) Find an example of a field k, a field extension L/k, and a finitely generated k-algebra R such that R is regular but $R \otimes_k L$ is not regular.

4.* Let R be a ring. Prove that the polynomial ring $R[x]$ in one indeterminant x with coefficients in R is a regular ring if R is a regular ring.
(Hint: You'll need to check that, for any prime ideal $\mathfrak{p} \subset R[x]$, the localization $R[x]_{\mathfrak{p}}$ of $R[x]$ at $\mathfrak{p}$ is a regular local ring. Fix one such prime $\mathfrak{p}$ and let $\mathfrak{q} = \mathfrak{p} \cap R$. Then $R[x]_{\mathfrak{p}}$ is a localization of $R_{\mathfrak{q}}[x]$ so it suffices to prove that $R_{\mathfrak{q}}[x]$ is regular. This reduces the proof to the case that R is a regular local ring. Now try to use the fact that $\mathrm{Kr.dim}(A[x]) = \mathrm{Kr.dim}(A) + 1$ for any Noetherian ring A from [AM69, Chapter 11, Exercise 7].)

5. Let k be a field, and let $R = k[x]/(x^2)$.
   (a) Prove that $K(R)$ is isomorphic with $\mathbb{Z}$.
   (b) Prove that $G(R)$ is also isomorphic with $\mathbb{Z}$.
   (c) Show that the homomorphism
   $$\varphi_R : K(R) \to G(R),$$
   induced by the inclusion $P_{fg}(R) \subset M_{fg}(R)$, has nontrivial cokernel.

6. Find a ring R such that the canonical homomorphism $\varphi_R : K(R) \to G(R)$ has nontrivial kernel $\ker(\varphi_R) \neq 0$.

7. In this exercise we prove an analog of Exercise 1.3.11 for rings of power series in one formal variable. Specifically, we show that if R is a Noetherian regular ring then there is a canonical isomorphism $\mathrm{Cl}(R) \cong \mathrm{Cl}(R[[x]])$.
   (a) Let R be any ring. Let $R[[x]]$ be the ring of formal power series over R in one formal variable x. Prove that the composition
   $$\mathrm{Pic}(R) \xrightarrow{\mathrm{res}_R^{R[[x]]}} \mathrm{Pic}(R[[x]]) \xrightarrow{\mathrm{res}_{R[[x]]}^R} \mathrm{Pic}(R),$$
   induced by the canonical inclusion $R \to R[[x]]$ and the canonical projection $R[[x]] \to R$ sending x to 0, is the identity.
   (b) Let R be any ring. Use both part (a) above, and part (e) of Exercise 1.2.8, to show that both of the restriction maps from part (a) are isomorphisms. (Hint: consider the inclusion $\mathrm{Pic}(S) \subset K(S)^{\times}$ of units for any ring S.)

(c) Now assume that R is both Noetherian and a regular ring. Prove that the ring of formal power series $R[[x]]$ with coefficients in R is then a Noetherian regular ring as well. Since a regular ring is locally factorial, Theorem 1.3.27 shows that there are isomorphisms

$$\mathrm{Cl}(R) \cong \mathrm{Pic}(R) \quad \text{and} \quad \mathrm{Cl}(R[[x]]) \cong \mathrm{Pic}(R[[x]]).$$

Together with parts (a) and (b), this proves the claim. See Exercise 1.3.10 for an explicit description of this isomorphism in terms of Weil divisors.

There exist examples of rings R which are UFDs, but which are not regular, such that the ring $R[[x]]$ is not a UFD. For example, if k is any field then one can take the ring $R = k(t)[[a, b, c]]/(a^2 + b^3 + tc^6)$ where t is an indeterminate. For more on this problem, see the notes of Lipman [Lip75].

8. Let R be a Dedekind domain with fraction field F and let $I \subset F$ be a fractional ideal for R. Prove that I is generated by at most two elements, i.e. show that there exist $f, g \in F$ so that $fR + gR = I$.
(Hint: analyze the unrelated claim in the proof of Lemma 1.5.44, along with its proof. Try to find f, g so that $fI^{-1} + gI^{-1} = R$.)

1.6 K-theory of a semisimple algebra

For use in future sections, we define here an analog of the K-group for an arbitrary associative, but possibly noncommutative, ring A. We then study these K-groups in detail in the more restrictive case that A is a *semisimple k-algebra* for k a field. Since in the study of algebraic geometry one typically focuses on rings which are commutative, we first recall the basic structure theorems on noncommutative rings which will be of use to us. One can find references to the results in this section in either the definitive [Row91] or the more leisurely [FD93].

We say that the ring A is *left-semisimple* if A is semisimple when considered as a left A-module under itself. So A is left-semisimple if there exists both a collection of irreducible left-ideals $I_1, ..., I_m$ (meaning that each of the ideals I_j themselves contain no proper left ideal of A) and an isomorphism of left A-modules

$$A \cong I_1 \oplus \cdots \oplus I_m.$$

We say that A is *right-semisimple* if it satisfies the analogous condition replacing left everywhere with right.

The *center of a ring* A, denoted $Z(A)$, is the collection of all elements $x \in A$ so that $xy = yx$ for all $y \in A$, i.e.

$$Z(A) = \{x \in A : xy = yx \text{ for all } y \in A\}.$$

The center is, in a natural way, a subring of A. If k is a field, then we say that A is a k-algebra to mean that there is a fixed ring homomorphism $k \to Z(A) \subset A$ which allows us to consider the action of multiplication by elements of k on A.

Theorem 1.6.1. *Fix a field k and let A be a finite dimensional and associative k-algebra. Then the following statements are true:*

(1) *A is left-semisimple if and only if A is right-semisimple.*
(2) *A is left-semisimple (resp. right-semisimple) if and only if every short exact sequence of left (resp. right) A-modules splits.*
(3) *A is left (or right)-semisimple if and only if A is isomorphic to a Cartesian product*
$$A \cong A_1 \times \cdots \times A_m$$
of simple k-algebras $A_1, ..., A_m$.

It therefore makes sense, when A is a finite dimensional and associative k-algebra, to say that A is semisimple without specifying whether it is left or right semisimple (which we now do). We also point out, in a conflict of terminology, that a ring A is *simple* if the only two-sided (both left and right) ideals $I \subset A$ are $I = 0$ and $I = A$. In particular, a simple ring A may not be simple as either a left, or as a right, module under itself (i.e. there may be left ideals, or right ideals, which are not simultaneously two-sided ideals).

Theorem 1.6.1 allows us to describe explicitly the structure of a left (or right) A-module M under in terms of a decomposition of A into a product of simple rings. Namely, any such decomposition of A induces a decomposition on the module M.

Corollary 1.6.2. *Suppose that $A \neq 0$ is a finite dimensional, associative, unital, and semisimple k-algebra. Decompose A as a product*

$$A \cong A_1 \times \cdots \times A_m$$

of nonzero simple k-algebras $A_1, ..., A_m$. Then every left (resp. right) A-module M can be decomposed (uniquely) into a product

$$M \cong M_1 \times \cdots \times M_m$$

with M_i a left (resp. right) A_i-module.

This means that the study of modules under a semisimple k-algebra A as in Theorem 1.6.1 reduces to the study of modules under certain simple k-algebras. For this we have the following structural result:

Theorem 1.6.3. *Fix a field k and let A be a finite dimensional, associative, and simple k-algebra. Then the following statements are true.*

(1) *The center $Z(A)$ of A is field extension F of k of finite degree $[F : k] < \infty$.*
(2) *There exists an F-algebra D with division (meaning that all nonzero elements in D have two-sided multiplicative inverses) and an isomorphism $A \cong M_n(D)$ between the ring A and the ring of $n \times n$-matrices with coefficients in D.*
(3) *Every simple left (resp. right) A-module M is isomorphic, as an A-module, to the direct sum $M \cong D^{\oplus n}$ considered with the canonical action by left (resp. right) multiplication compatible with a fixed isomorphism $A \cong M_n(D)$. Moreover, every left (resp. right) A-module M is isomorphic to a sum of simple left (resp. right) A-modules.*

To summarize: an associative, semisimple k-algebra A having finite dimension as a k-vector space admits a finite collection of finite extensions $F_1/k, ..., F_m/k$ and an isomorphism

$$A \cong M_{n_1}(D_1) \times \cdots \times M_{n_m}(D_m)$$

with division F_i-algebras D_i for $i = 1, ..., m$; moreover any such decomposition completely determines the structure of all left or right A-modules.

Example 1.6.4. If k is an algebraically closed field, then there are no finite dimensional division algebras over k other than k itself. That is, if D is a finite dimensional division k-algebra then any nonzero $d \in D$ defines ring homomorphism

$$\phi_d : k[x] \to D$$

uniquely determined by the condition that $\phi_d(x) = d$. The kernel of ϕ_d is a nonzero ideal (by the finiteness of the dimension of D over k) which is moreover maximal (since D is a division k-algebra). Since k is algebraically closed, this ideal is of the form $\ker(\phi_d) = (x - c)$ for some element $c \in k$. Hence $d = c \in k$ as well.

Consequently, if k is an algebraically closed field then any simple and associative k-algebra A of finite dimension and with center $Z(A) = k$ is isomorphic to a matrix ring $M_n(k)$ for some $n \in \mathbb{N}$. Conversely, if k is an arbitrary field and if A is an associative and finite dimensional k-algebra, then there is a natural number n and a field extension F/k admitting an isomorphism

$$A \otimes_k F \cong M_n(F)$$

if and only if $A \cong M_m(D)$ for some division k-algebra D with center $Z(D) = k$. Algebras of this form are called *central simple k-algebras.*

In the setting of noncommutative rings, the definition of a projective module still makes sense so long as one specifies whether the action of the coefficient ring is via the left or the right. So it makes sense to define, nearly verbatim, the K-theory of a noncommutative ring as follows.

Definition 1.6.5. Let A be an arbitrary associative ring. Let $P_{fg,l}(A)$ be the free abelian group on isomorphism classes of finitely generated projective left A-modules, i.e. let

$$P_{fg,l}(A) := \bigoplus_M \mathbb{Z} \cdot M$$

where the index M varies over the choice of a representative for each isomorphism class of finitely generated projective left A-module. Let $P_{ex,l}(A) \subset P_{fg,l}(A)$ be the subgroup generated by elements $M - L - N$ for each short exact sequence

$$0 \to L \to M \to N \to 0$$

of finitely generated projective left A-modules L, M, and N. We define the *left K-theory of the ring A* as the quotient group $K(A) = P_{fg,l}(A)/P_{ex,l}(A)$.

We could similarly define the *right K-theory of A* by replacing everywhere the word left appears in the above definition with the word right instead. Then the right K-theory of A would be canonically isomorphic to the left K-theory $K(A^{op})$ of the *opposite ring A^{op} of A* (i.e. the ring which has the same underlying abelian group as A but with multiplication $x \circ y = yx$).

Theorem 1.6.6. *Fix a field k and let $A \neq 0$ be an associative, semisimple, and finite dimensional k-algebra with a decomposition*

$$A \cong A_1 \times \cdots \times A_m$$

into a product of finitely many nonzero simple k-algebras $A_1, ..., A_m$. Then the left K-theory of A decomposes

$$K(A) \cong K(A_1) \times \cdots \times K(A_m)$$

accordingly.

Moreover, if $A \neq 0$ is an associative, simple, and finite dimensional k-algebra then there is a natural isomorphism

$$\mathbb{Z} \cong K(A)$$

under which the generator 1 of $\mathbb{Z}$ is mapped to the isomorphism class $[M]$ of a simple left A-module M.

Proof. We prove the second statement first. Suppose that A is a simple k-algebra so, by Theorem 1.6.3, there is a finite field extension F/k, a division F-algebra D, and an isomorphism $A \cong M_n(D)$. Every left A-module M is isomorphic then to a sum of copies of the simple module $D^{\oplus n}$ and, if M is finitely generated, only finitely many such copies may occur. By Theorem 1.6.1 (2) all short exact sequences of A-modules split, so $K(A) \cong \mathbb{Z}$ is additively generated by the class $[D^{\oplus n}]$.

For the first statement, let M be any left A-module where now A is assumed to be an arbitrary semisimple k-algebra. According to Corollary 1.6.2 we can write $M \cong M_1 \times \cdots \times M_m$ for some left A_1, ..., A_m-modules M_1, ..., M_m respectively. For each integer i with $1 \leq i \leq m$, there is then a canonical map

$$K(A) \to K(A_i)$$

given by projecting from M to M_i. If M_i is a simple A_i-module, then M_i is also a finitely generated and projective A-module. The associated map to the product

$$K(A) \to K(A_1) \times \cdots \times K(A_m)$$

must therefore be an isomorphism since $K(A)$ is generated by the classes of simple A_i-modules over all varying i with $1 \leq i \leq m$. □

Exercises for Section 1.6

1. Let $Q = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}ij$ be a 4-dimesional $\mathbb{R}$-vector space with basis the elements $1, i, j, ij$. Define an $\mathbb{R}$-algebra structure on Q by letting 1 act as a two-sided identity and with further relations

$$i^2 = -1, \quad j^2 = -1, \quad \text{and} \quad ij = -ji.$$

Show that Q is an associative $\mathbb{R}$-algebra by constructing an isomorphism of $\mathbb{C}$-algebras $Q \otimes_{\mathbb{R}} \mathbb{C} \cong M_2(\mathbb{C})$. Show that Q is a division $\mathbb{R}$-algebra by considering for any element $x = a + bi + cj + dij$ the element $\bar{x} = a - bi - cj - dij$.

2. Let k be a field and let D be a finite dimensional associative division k-algebra. Show that for any $n \geq 1$ the ring $M_n(D)$ is left-semisimple and that any simple left $M_n(D)$-module is isomorphic with $D^{\oplus n}$ by utilizing the following argument.
   (a) Prove the claim directly when $n = 1$. If M is a simple left D-module, then consider a left submodule $Dx \subset M$ generated by an element $x \in M$.
   (b) For $n > 1$, consider the set $I_r \subset M_n(D)$ of matrices with all potentially nonzero elements contained only in the rth column and with zeros everywhere else; show that I_r is a left-ideal.
   (c) Let $0 \neq M \in I_r$ be an arbitrary element. Show that M generates I_r as a left $M_n(D)$-module; hence I_r is a simple $M_n(D)$-module for any $1 \leq r \leq n$. (Hint: consider the products by elementary matrices $E_{i,r}M$ where $E_{i,j} = (\delta_{ik} \cdot \delta_{lj})_{1 \leq k,l \leq n}$ and $\delta_{*,*}$ is the Kronecker delta function.)

(d) Show that there is an isomorphism of $M_n(D)$-modules

$$\bigoplus_{1\leq r\leq n} I_r \cong M_n(D),$$

thereby proving that $M_n(D)$ is a semisimple k-algebra. Also, if N is any other simple left $M_n(D)$-module, then consider the morphism

$$\bigoplus_{1\leq r\leq n} I_r \cong M_n(D) \to N, \quad x \mapsto xy$$

for a fixed element $0 \neq y \in N$ to show that $I_r \cong N$ for some $1 \leq r \leq n$. Hence all simple $M_n(D)$-modules are isomorphic with $I_r \cong D^{\oplus n}$.

K-Theory of Schemes

G-Theory of Schemes I

3.1 Properties of the G-functor

3.2 Localization

Example 3.2.1. Something something

Example 3.2.2. something

3.3 Applications of dévissage

3.4 Applications of resolution

3.5 The topological filtration

G-Theory of Schemes II

The Adams–Grothendieck–Riemann–Roch Theorem

The K-Theory of Forms

6.1 K-theory of Severi–Brauer varieties

6.2 K-theory of smooth quadrics

Applications

7.1 Sums of squares formulas

7.2 Torsion algebraic cycles

Spectral Sequences

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Notation

$\mathbb{N}$	the set of natural numbers $\{1, 2, 3, ...\}$
$\mathbb{Z}_{\geq 0}$	the set of nonnegative integers $\mathbb{N} \cup \{0\}$
$\#S$	the cardinality of a set S
k	a field
R	an associative, commutative, and unital ring
F	the fraction field $F = R_{(0)}$ of a domain R
$\mathrm{ht}(\mathfrak{p})$	the height of a prime ideal
$\mathrm{Kr.dim}(R)$	the Krull dimension of a ring R
$\mathrm{pd}_R(M)$	the projective dimension of an R-module M
A	an associative and unital ring
A^{op}	the opposite ring of A
$R^\times$	the group of units of a ring R
R^ν	the integral closure of a domain R in its field of fractions
$\mathrm{Ann}_R(M)$	the R-module annihilator of M
$\wedge^i M$	i-th exterior power of M
$P_{fg}(R)$	free abelian group on isomorphism classes of finitely generated projective R-modules
$P_{ex}(R)$	subgroup of $P_{fg}(R)$ from short exact sequences
$P_{lex}(R)$	subgroup of $P_{fg}(R)$ from long exact sequences
$K(R)$	the K-theory of the ring R, i.e. $P_{fg}(R)/P_{ex}(R)$
$M_{fg}(R)$	free abelian group on isomorphism classes of finitely generated R-modules
$M_{ex}(R)$	subgroup of $M_{fg}(R)$ from short exact sequences
$G(R)$	the G-theory of the ring R, i.e. $M_{fg}(R)/M_{ex}(R)$
$P_{fg,l}(A)$	free abelian group on isomorphism classes of finitely generated projective left A-modules
$P_{ex,l}(A)$	subgroup of $P_{fg,l}(A)$ from short exact sequences
$K(A)$	the left K-theory of the ring A, i.e. $P_{fg,l}(A)/P_{ex,l}(A)$
$X^{(r)}$	the set of prime ideals of height r if $X = \mathrm{Spec}(R)$
$X^{(r)}$	the set of codimension-r points of a scheme X

$X_{(r)}$	the set of dimension-r points of a scheme X
$Z^n(R)$	group of height n-cycles on R
$Z(R)$	group of all cycles on R
$\partial Z(R)$	the kernel of the map $\mathrm{cl} : Z(R) \to G(R)$
$\mathrm{Pic}(R)$	the group of invertible R-modules
$\det$	the determinant (map, or of a module)
$I_{fr}(R)$	the group of invertible fractional ideals
$I_{pr}(R)$	the group of principal fractional ideals
$\mathrm{ord}_{\mathfrak{p}}$	the order of vanishing
$\mathrm{div}(f)$	the associated Weil divisor of a rational function f
$\mathrm{div}(I)$	the associated Weil divisor of a fractional ideal I
$\mathrm{WDiv}(R)$	the group of Weil divisors of R
$\mathrm{Cl}(R)$	the ideal class group of R, i.e. $\mathrm{WDiv}(R)/\mathrm{div}(F^\times)$
res_R^S	a restriction map from something associated with R to S
$e(\mathfrak{q}/\mathfrak{p})$	the ramification index of $\mathfrak{q}$ over $\mathfrak{p}$
c_1	the first Chern class

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