Finding noncyclic torsion in $CH^2(X)$ when X is a Severi-Brauer variety

Notation:

- k a base field
- A a finite dim. CSA$/_k$
- $X = SB(A)$ the Severi-Brauer variety of A
- $CH^i(X)$ the group of codim.-i cycles on X mod $\sim_{rat}$
- $K(X)$ $(\cong G(X))$ the Grothendieck ring of loc. free sheaves (group of coh. sheaves)

Initial Problem: What are the groups $CH^i(X)$?

Some History:

0) $CH^i(X) \otimes \mathbb{Q} \cong \mathbb{Q}$ for $0 \leq i \leq \dim(X)$

1) $CH^i(X)$ can contain nontrivial torsion for $i \geq 3$ [Merkurjev '95]

2) $CH^2(X)$ can contain cyclic torsion [Karpenko '95]

3) $CH^i(X)$ can contain non cyclic torsion [Karpenko '16]
(example with $i=10$)

Theorem: There exists $X = SB(A)$ with $CH^2(X)_{tors}$ containing a noncyclic group. [M. '20]

Ex The smallest example with 2-primary index has $ind(A) = 2^5$ and $CH^2(X) = \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$.

Part 1:

the main ideas

Idea for Proof: use $F_\gamma^i(X) \subset K(X)$ and $F_\tau^i(X) \subset G(X)$.

More specifically, use that there are maps

$$\begin{array}{ccccc} gr_\gamma^2 K(X) & \xrightarrow{\phi^2} & gr_\tau^2 G(X) & \xleftarrow{\sim} & CH^2(X). \\ [\mathcal{F}] & \longmapsto & [\mathcal{F}] \quad \text{and} & [\mathcal{O}_V] & \longmapsfrom \ V \end{array}$$

The group $gr_\gamma^2 K(X)$ depends only on $r\mathrm{Bch}(A)$, and for each of these there is a "generic" X with ϕ^2 an iso.

So just need to find an X with $gr_\gamma^2 K(X)_{tors}$ noncyclic.

Can also reduce to the case $\mathrm{ind}(A) = p^n$ for a prime p.
Then it's known that the λ-ring $K(X)$ is generated by $\mathrm{lev}(A)+1$ $\left(\leq \lfloor \frac{n}{2} \rfloor + 1\right)$ many bundles $\zeta_X(1), \ldots, \zeta_X(p^{ik})$.

[So, if $\lfloor \frac{n}{2} \rfloor + 1 = 3$, can find X with $gr^2_\gamma K(X)$ having at most 3 generators.]

But, it's still difficult to compute their relations integrally.

New idea: work with $\mathbb{F}_p$-coefficients, e.g. in $K(X) \otimes \mathbb{F}_2$.

Def$^{\underline{n}}$: for a ring S, define th

S-γ-filtration $F^i_{\gamma,S}(X) = \mathrm{Im}\left(F^i_\gamma(X) \otimes S \to K(X) \otimes S\right)$

and S-topological-filtration $F^i_{\tau,S}(X) = \mathrm{Im}\left(F^i_\tau(X) \otimes S \longrightarrow G(X) \otimes S\right)$.

For $X = SB(A)$, we have $\dim_{\mathbb{F}_p} K(X) \otimes \mathbb{F}_p = \deg(A) = \sum_{i \in \mathbb{Z}} \dim_{\mathbb{F}_p}\left(\dfrac{F^i_{\tau,\mathbb{F}_p}(X)}{F^{i+1}_{\tau,\mathbb{F}_p}(X)}\right)$.

Thm: Let Y be a regular variety. Assume $G(Y)$ is $\mathbb{Z}$-flat. Then, for any ring S, there are isomorphisms

(codim 0) $$gr^0_{\tau} G(X) \otimes S \cong F^0_{\tau,S}(X)/F^1_{\tau,S}(X) \cong S$$

(codim 1) $$gr^1_{\tau} G(X) \otimes S \cong F^1_{\tau,S}(X)/F^2_{\tau,S}(X) \cong \mathrm{Pic}(X) \otimes S$$

(codim 2) $$gr^2_{\tau} G(X) \otimes S \cong F^2_{\tau,S}(X)/F^3_{\tau,S}(X) \cong CH^2(X) \otimes S.$$

Part 2:

the computations

The Example: A is "generic" with $\mathrm{ind}(A) = 2^5$, $\mathrm{ind}(A^{\otimes 2}) = 2^3$, and $\mathrm{ind}(A^{\otimes 4}) = 1$, i.e. $\mathrm{rBeh}(A) = (5, 3, 0)$.

$K(X) \otimes \mathbb{F}_2 = \bigoplus_{i=0}^{31} \mathbb{F}_2 \cdot v_i$ where $v_i = [\zeta_X(i)]$ and the $\zeta_X(i)$ are defined by $\zeta_X(i) = \zeta_X^{\otimes i} \otimes_{A^{\otimes i}} M_i$ with M_i a simple left $A^{\otimes i}$-module and $\zeta_X = \pi^* S_{Gr}$ under the embedding

$$X \subset Gr(\deg(A), A).$$

Computing $F^i_{\gamma,\mathbb{F}_2}(X) = F^i_{\tau,\mathbb{F}_2}(X)$:

literally write down all elements in the filtration. Generators are the images of products

$$\psi(r,s,t) = \prod_{i=1}^{r} \gamma^{a_i}(\xi_X(1)-32) \cdot \prod_{j=1}^{s} \gamma^{b_j}(\xi_X(2)-8) \cdot \prod_{k=1}^{t} \gamma^{c_k}(\xi_X(4)-1).$$

Can compute $\gamma^i(\xi_X(j)-2^{n_j})$ integrally and structure constants are given by $v_i \cdot v_j = 0$ unless $4|i$ or $4|j$ where $v_i \cdot v_j = v_{i+j}$.

Example:

Here $x_i = \gamma^i(\zeta_X(1) - 32)$

$y_i = \gamma^i(\zeta_X(2) - 8)$

$z_i = \gamma^i(\zeta_X(4) - 1)$

Find torsion in codimensions 2, 4, 6, 8, 10, 12, 14 and noncyclic torsion in codimensions 2 and 10.

TABLE 6. $r\mathcal{B}eh(A) = (5,3,0)$

i	$\dim_{\mathbb{F}_2}\left(\gamma_{\mathbb{F}_2}^{i/i+1}(X)\right)$	generators	$\sum_{j\le i}\dim_{\mathbb{F}_2}\left(\gamma_{\mathbb{F}_2}^{j/j+1}(X)\right)$
0	1	ν_0	1
1	1	z_1	2
2	3	x_2, y_2, z_1^2	5
3	1	z_1^3	6
4	2	x_4, y_4	8
5	0	-	8
6	2	x_6, y_6	10
7	1	y_7	11
8	2	x_8, y_8	13
9	1	$y_8 z_1$	14
10	3	$x_{10}, y_2 y_8, y_8 z_1^2$	17
11	1	$y_8 z_1^3$	18
12	2	$x_{12}, y_4 y_8$	20
13	0	-	20
14	2	$x_{14}, y_6 y_8$	22
15	1	$y_7 y_8$	23
16	1	x_{16}	24
17	0	-	24
18	1	x_{18}	25
19	0	-	25
20	1	x_{20}	26
21	0	-	26
22	1	x_{22}	27
23	0	-	27
24	1	x_{24}	28
25	0	-	28
26	1	x_{26}	29
27	0	-	29
28	1	x_{28}	30
29	0	-	30
30	1	x_{30}	31
31	1	x_{31}	32

Torsion subgroups of $CH^2(X)$

TABLE 1. For generic algebras of index 8

	$r\mathcal{B}eh(A)$	$\mathrm{lev}(A)$	$Q^2(X)$	$\mathrm{Tor}_1(\mathbb{Q}/\mathbb{Z}, \mathrm{CH}^2(X))$
1	$(3,2,1,0)$	0	0	0
2	$(3,2,0)$	1	0	0
3	$(3,1,0)$	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
4	$(3,0)$	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$

TABLE 2. For generic algebras of index 16

	$r\mathcal{B}eh(A)$	$\mathrm{lev}(A)$	$Q^2(X)$	$\mathrm{Tor}_1(\mathbb{Q}/\mathbb{Z}, \mathrm{CH}^2(X))$
1	$(4,3,2,1,0)$	0	0	0
2	$(4,3,2,0)$	1	0	0
3	$(4,3,1,0)$	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
4	$(4,3,0)$	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
5	$(4,2,1,0)$	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
6	$(4,2,0)$	2	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
7	$(4,1,0)$	1	$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
8	$(4,0)$	1	$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$

TABLE 3. For generic algebras of index 32

	$r\mathcal{B}eh(A)$	$\mathrm{lev}(A)$	$Q^2(X)$	$\mathrm{Tor}_1(\mathbb{Q}/\mathbb{Z}, \mathrm{CH}^2(X))$
1	$(5,4,3,2,1,0)$	0	0	0
2	$(5,4,3,2,0)$	1	0	0
3	$(5,4,3,1,0)$	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
4	$(5,4,3,0)$	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
5	$(5,4,2,1,0)$	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
6	$(5,4,2,0)$	2	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
7	$(5,4,1,0)$	1	$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$
8	$(5,4,0)$	1	$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$
9	$(5,3,2,1,0)$	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
10	$(5,3,2,0)$	2	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
11	$(5,3,1,0)$	2	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$
12	$(5,3,0)$	2	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$
13	$(5,2,1,0)$	1	$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
14	$(5,2,0)$	2	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
15	$(5,1,0)$	1	$\mathbb{Z}/8\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
16	$(5,0)$	1	$\mathbb{Z}/8\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$

Conjecture: For any prime p,

$$\max_{\mathrm{lev}(A)=n} \left\{ \dim_{\mathbb{F}_p} CH^2(SB(A)) \otimes \mathbb{F}_p \right\} = n+1$$

where the max is taken over all A (over all k) with $\mathrm{ind}(A) = p^m$ for some $m > 0$ and $\mathrm{lev}(A) = n$.

Question: What do $CH^2(SB_r(A))$ look like?

Thanks