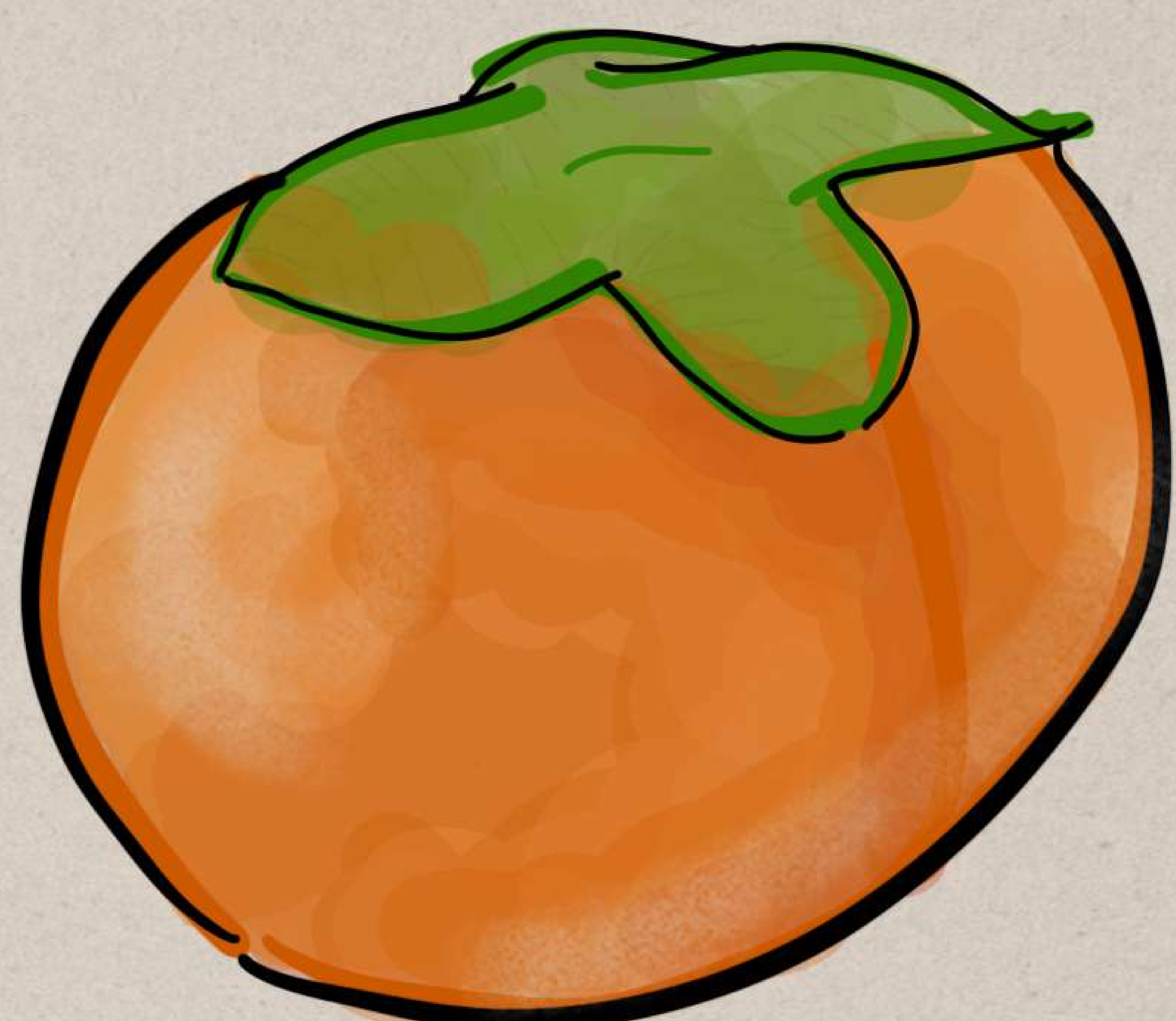
On the Chow groups of Severi-Brauer varieties associated to biquaternion algebras

(i.e. computing $CH_1(SB(Q_1 \otimes Q_2))$)



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Question: What do the Chow groups of X look like?
Can we compute these groups for specific X (i.e. for specific A)?

Part 1:

some background

Objects:

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Rmk: any central simple k-algebra A has the property that $A \otimes_k \bar{k} \cong M_n(\bar{k})$.

Let A be a CSA/k.

Def$^{\underline{n}}$: The Severi-Brauer variety associated to A is the subvariety

$$X = SB(A) \subseteq Gr(n, A) \qquad \text{with } n = \sqrt{\dim A}$$

whose R-points are

$$\begin{aligned} X(R) &= \{ I \subset A \otimes R : I \text{ is a right ideal} \} \\ &\subseteq \{ I \subset A \otimes R : I \text{ is a projective summand of } \operatorname{rank}(I) = n \} \\ &= Gr(n, A)(R). \end{aligned}$$

Ex: If $A = M_n(k)$ then there is a correspondence

$$\left\{ \begin{array}{c} \text{right ideals } I \subset A \\ \text{of } \mathrm{rank}(I) = n \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{1-dimensional subspaces} \\ \text{of } k^{\oplus n} \end{array} \right\}$$

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$$I \longmapsto \begin{array}{c} \text{the span of the columns of} \\ \text{elements of } I \end{array}$$

$$\begin{array}{c} \text{the set of matrices with} \\ \text{columns in } L \end{array} \longleftarrow\!\!\!\mid \; L \subset k^{\oplus n}$$

(If $A = (v_1 \cdots v_n)$ with $v_1, \dots, v_n \in L$ and if $M = (w_1 \cdots w_n)$ then $AM = (Aw_1 \cdots Aw_n)$ and each $Aw_i \in \operatorname{Span}\{v_1, \dots, v_n\} \subseteq L$.)

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$$SB(M_n(k)) \cong \mathbb{P}^{n-1}.$$

Rmk: Since $A \otimes \bar{k} \cong M_n(\bar{k})$ for any $CSA_{/k}$ A, we have that $SB(A) \times_k \bar{k} \cong SB(M_n(\bar{k})) \cong \mathbb{P}^{n-1}_{\bar{k}}$.

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If $\mathrm{char}(k) \neq 2$ then any quaternion algebra Q/k has a presentation $Q = k \oplus ki \oplus kj \oplus kij$ where

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The Severi-Brauer variety of Q is isomorphic with a smooth conic in $\mathbb{P}^2$.

If $X = SB(Q)$, then $X \times_k \bar{k} \cong \mathbb{P}^1_{\bar{k}}$. The line bundle T_X then realizes this embedding. Can arrange $X \cong V(ax^2 + by^2 - abz^2) \subset \mathbb{P}^2$.

Ex (cont'd): If $k = \mathbb{R}$ and $a = -1$, $b = -1$ then get Hamilton's quaternions $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}ij$ with

$$i^2 = -1, \quad j^2 = -1, \quad \text{and} \quad ij = -ji.$$

Here $X = SB(\mathbb{H}) \cong V(-x^2 - y^2 - z^2) \subset \mathbb{P}^2$.

Since $X(\mathbb{R}) = \emptyset$, there are no isomorphisms between X and $\mathbb{P}^1$ over $\mathbb{R}$, i.e. X is nontrivial.

Part 2:

the problem and its history

Problem: A Severi-Brauer variety X defined over k is isomorphic to $\mathbb{P}^n_F$ after extending scalars X_F to a finite extension F/k. A restriction-correstriction argument shows that $CH_i(X)$ sits in an exact sequence

$$0 \to K_i \to CH_i(X) \to CH_i(\mathbb{P}^n_F) = \begin{cases} \mathbb{Z} & \text{if } 0 \le i \le n \\ 0 & \text{otherwise} \end{cases}$$

and $[F:k] \cdot K_i = 0$. What can be said about K_i?

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(3) if $X = SB(A)$ and $A = Q_1 \otimes Q_2$ is biquaternion, then $CH_1(X) = CH^2(X)$ is torsion-free

[Karpenko, '96]

(4) $CH^2(X)$ can contain non-trivial cyclic torsion [Karpenko, '98]

History:

(5) $CH_0(X)$ is always torsion-free

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(6) Recent computations of $CH(X)$ when X is "generic"

[Karpenko '16, M.]

Recently, there's been renewed interest in computing $CH_i(X)$ for a general Severi-Brauer variety X.

All known methods for computing this group rely on K-theory:

- Quillen determined the ring structure of $K(X)$.

- the GRR thm (without denominators) compares $CH(X)$ to a graded ring associated with a filtration on $K(X)$.

- works well to describe $CH(X)$ if X is "generic" but not for general X.

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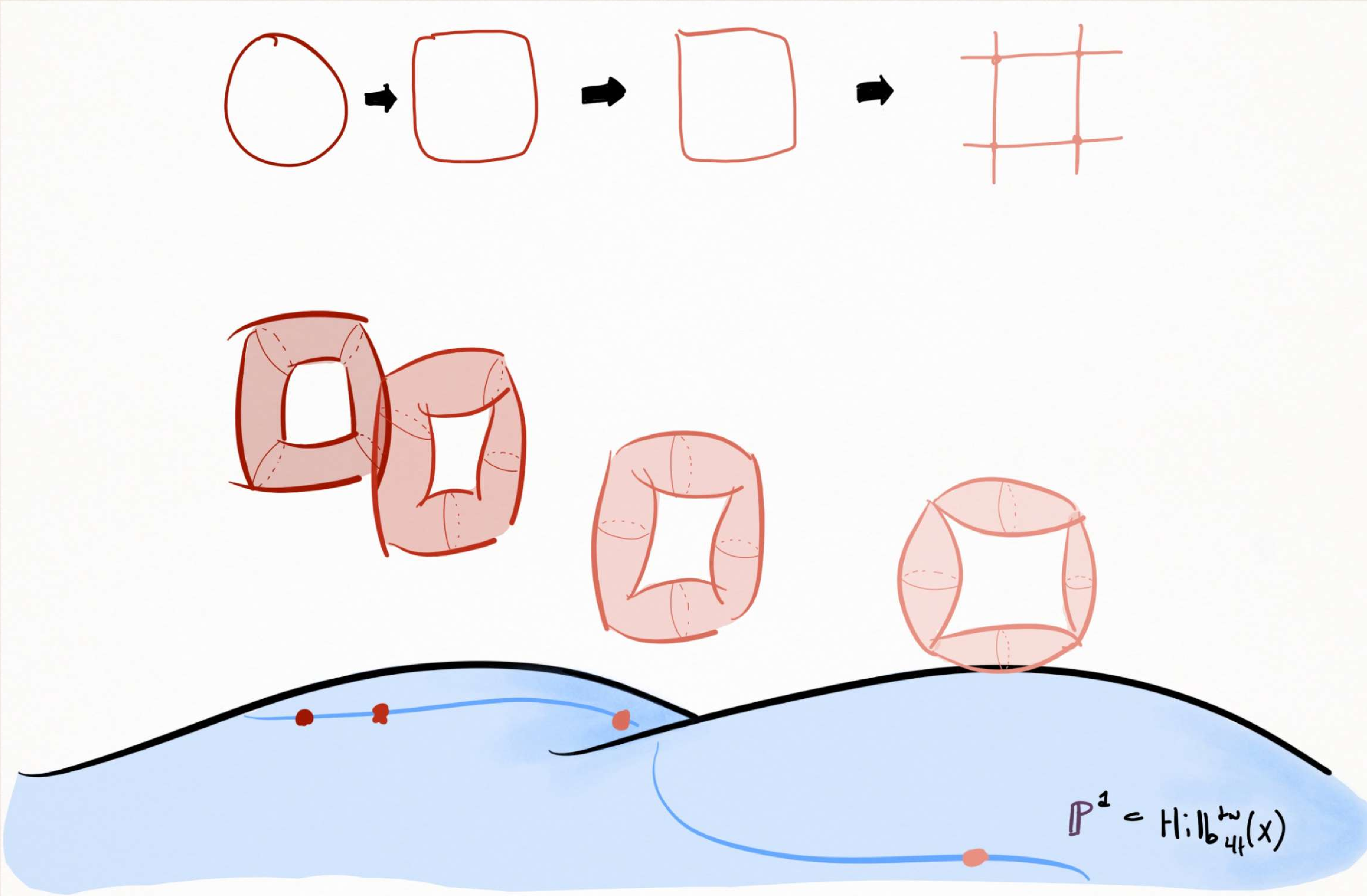
"Sketch of Proof": Identify generators η and ζ for $CH_1(X)$ with $\eta = [V]$ where $V_{\bar{k}}$ is the union of two skew lines in $\mathbb{P}^3$, and $\zeta = [E]$ with $E_{\bar{k}}$ an elliptic normal curve. Then show

$$2\eta = \zeta$$

by degenerating E to a geometrically reducible curve rep. 2η. $\varnothing$


$\mathbb{P}^1 \subset \mathrm{Hilb}^{tw}_{4t}(X)$

Part 3:

the proof

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Then Quillen's result shows $K(X) = \mathbb{Z} \cdot \mathcal{O}_X \oplus \mathbb{Z} \cdot \zeta_X(1) \oplus \mathbb{Z} \cdot \zeta_X(2) \oplus \mathbb{Z} \zeta_X(3)$

where $\zeta_X(1) = \pi^* \mathcal{S}_{Gr}$ is the pullback of the universal subbundle

under $\pi : X \hookrightarrow Gr(4, A)$, and $\zeta_X(i) = \zeta_X(1)^{\otimes i} \underset{A^{\otimes i}}{\otimes} M_i$

with M_i a simple left $A^{\otimes i}$-module.

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Have $rk(\zeta_X(1)) = 4$, $rk(\zeta_X(2)) = 1$, and $rk(\zeta_X(3)) = 4$.

Also, $\zeta_X(i) \times_k \bar{k} = \mathcal{O}(-i)^{rk(\zeta_X(i))}$. Hence $\zeta_X(3) = \zeta_X(1) \cdot \zeta_X(2)$.

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Using that $\zeta_X(3) = \zeta_X(1)\cdot\zeta_X(2)$, can just use:

$$c_1(\zeta_X(1))^2,\ c_2(\zeta_X(1)),\ c_1(\zeta_X(2))^2,\ c_1(\zeta_X(1))\cdot c_1(\zeta_X(2)).$$

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Since $\mathrm{Pic}(X) = \mathbb{Z}\cdot c_1(\zeta_X(2))$.

Generators:

$$3c_2(\xi_x(1)) - c_1(\xi_x(1))^2$$
$$= 3c_2(\xi_x(1)) - 4c_1(\xi_x(2))^2$$

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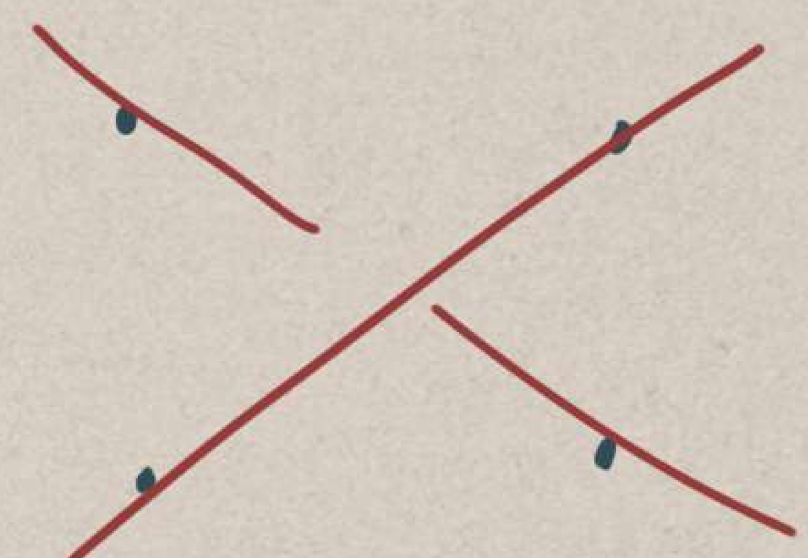
By specialization. Can pick a degree 4-pt on the generic SB-variety X^{gen} of index = exp = 4, find $V' \subset X^{gen}$ passing through this pt like



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$$c_1(\xi_X(2))^2 = [E]$$

By Bertini's thm, this is represented by the smooth intersection of two smooth quadric surfaces.

Get a geometrically elliptic normal curve $E \subset X$.

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(1) $\mathrm{Hilb}^{tw}_{\phi(t)}(X)(V) = \left\{ W \subseteq X \times V : \begin{array}{l} W \text{ is flat + proper over } V \text{ with} \\ \text{Hilb. poly. } \phi(t) \text{ over } \overline{k} \end{array} \right\}$

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(2) $\mathrm{Hilb}^{tw}_{\phi(t)}(X) \times_k \bar{k} \cong \mathrm{Hilb}_{\phi(t)}(\mathbb{P}^n_{\bar{k}})$

(3) If $\mathcal{L}$ is very ample on X of deg. m then

$$\mathrm{Hilb}^{tw}_{\phi(t)}(X) \cong \mathrm{Hilb}_{\phi(mt)}(X).$$

Relations: Use "twisted" Hilbert schemes $\mathrm{Hilb}^{tw}_{\phi(t)}(X)$.

$X = SB(Q_1 \otimes Q_2)$, $\phi(t) = 4t$ Can define a map

$$\Lambda : Gr(2, H^0(\mathbb{P}^3, \mathcal{O}(2))) \dashrightarrow \mathrm{Hilb}_{4t}(\mathbb{P}^3)$$

$$\langle f, g \rangle \longmapsto V(f) \cap V(g).$$

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$X = SB(Q_1 \oplus Q_2)$, $\phi(t) = 4t$ Can define a map

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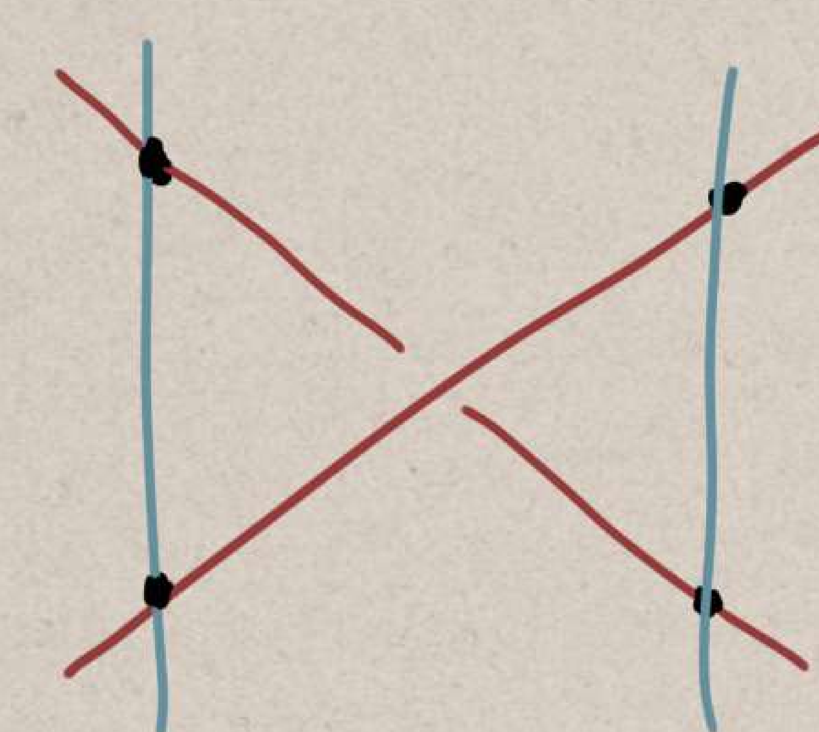
Contains the geom. elliptic normal curve E with $[E] = c_1(\mathcal{J}_X(2))^2$.

Also contains a curve that geom. looks like:



where the vertices are some degree 4-pt. on X.

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This last curve is the union of "two pairs of geom. skew lines".

Can also describe the twisted Hilbert scheme param. these.

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And $V_1 \cong Bl_\Delta(S^2(Gr(2,10)))$. [Chen, Coskun, Nollet, '11]

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This descends to a component of $\mathrm{Hilb}^{tw}_{2t+2}(X)$ iso. to $\mathrm{Bl}_\Delta(S^2(SB_2(A)))$.

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$X = SB(Q_1 \otimes Q_2)$, $\phi(t) = 2t+2$ $\quad Bl_0(S^2(SB_2(A))) \overset{\text{open}}{\supseteq} \acute{e}t_2(A)$ and

$\acute{e}t_2(A)$ param. those subschemes of X that are geo. 2 skew lines.

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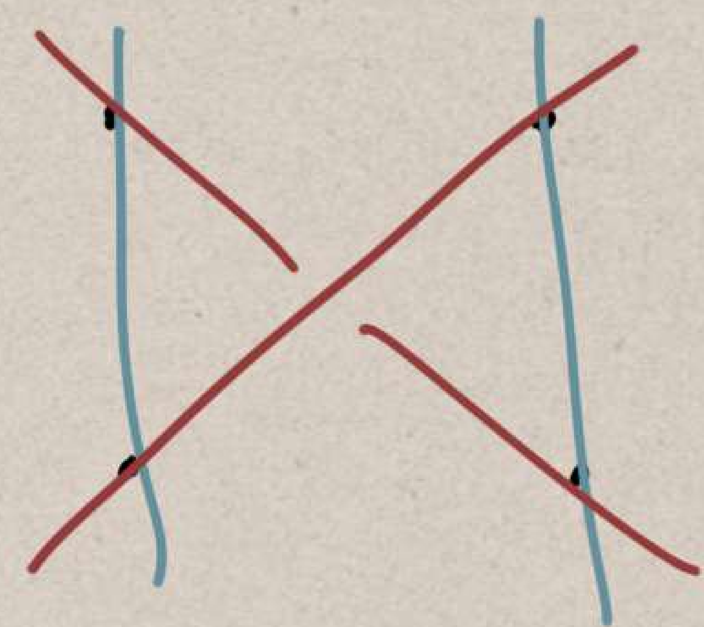
Thm: [Krashen '10, M.] The scheme $\acute{e}t_2(A)$ is R-trivial. I.e.

any two points can be connected via a map $\mathbb{P}^1 \dashrightarrow \acute{e}t_2(A)$.

Relations: **Step 1:** can degenerate $[E] = c_1(\xi_X(2))^2$ to a curve



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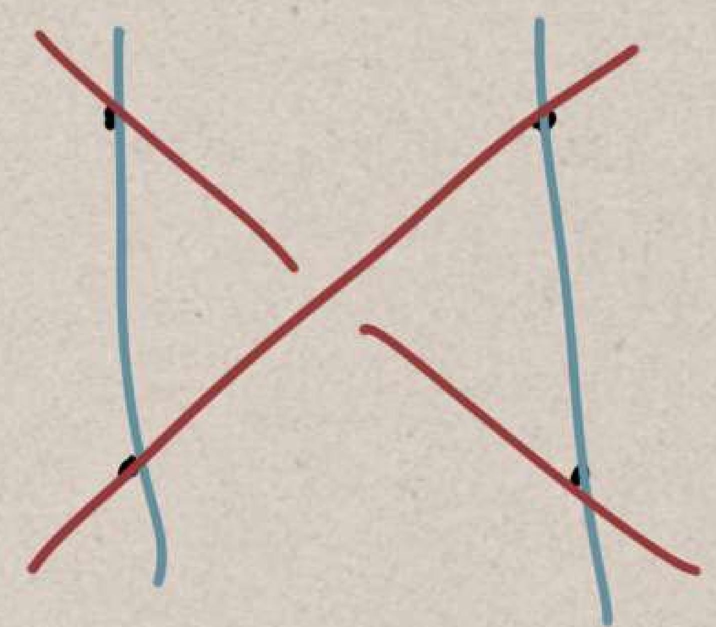

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Step 2: By R-triviality of $\acute{e}t_2(A)$ can connect

$$[C_1] = [C_2] = [V] = 3c_2(\zeta_X(1)) - c_1(\zeta_X(1))^2 = 3c_2(\zeta_X(1)) - 4c_1(\zeta_X(2))^2$$

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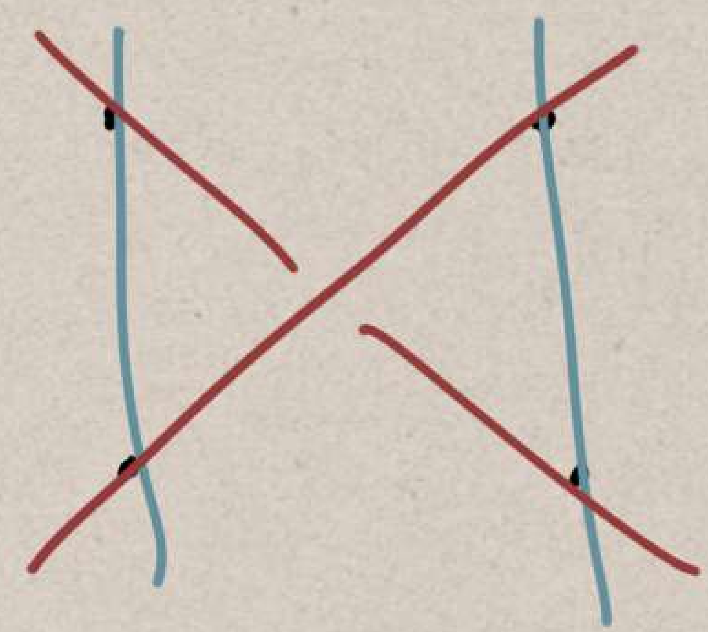

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Step 3: Can compute $CH_1(X) = \mathbb{Z}$ using

$$2(3c_2(\xi_X(1)) - c_1(\xi_X(1))^2) = [C_1 \cup C_2] = [C_1] + [C_2] = [E] = c_1(\xi_X(2))^2.$$

(Explicitly) Let $a = c_2(\xi_X(1))$ and $b = c_1(\xi_X(2))^2$, So $CH_1(X) = \underset{\sim}{\underline{\mathbb{Z}a \oplus \mathbb{Z}b}}$.

Then $CH_1(X) = \mathbb{Z}x$ where $x = 3a - 4b$. Easier to show $CH_1(X) \otimes \mathbb{Z}_{(2)} \cong \mathbb{Z}_{(2)} \cdot x$.

We showed $2x = b$. So $2x = 6a - 8b = b$ $\Rightarrow$ $6a = 9b$

$\Rightarrow$ $2a = 3b$ (using $\frac{1}{3} \in \mathbb{Z}_{(2)}$).

Now $x = 3a - 4b = 3a - b - 3b = 3a - b - 2a = a - b$.

So $x + 2x = a - b + b = a$. Altogether shows that x generates $CH_1(X) \otimes \mathbb{Z}_{(2)}$.

But $CH_1(X)$ has 2-primary torsion. $\cancel{\square}$

Thanks
again.