(Naive) $\mathbb{A}^1$-homotopy equivalences

and

Theorems of Whitehead and Zariski

Notation: $X, Y, Z /_k$ for varieties over an arbitrary field k

(often X, Y, Z will be smooth, sometimes proper or projective)

Goals: approach $\mathbb{A}^1$-homotopy theory from a very low-tech point of view and derive some basic results in this setting

(Feel free to ask questions at any time)

Part 1 :

naive $\mathbb{A}^1$-homotopy theory

Starting from somewhere simple ...

<u>Idea</u>: define two morphisms $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ to be "$\mathbb{A}^1$-homotopic"

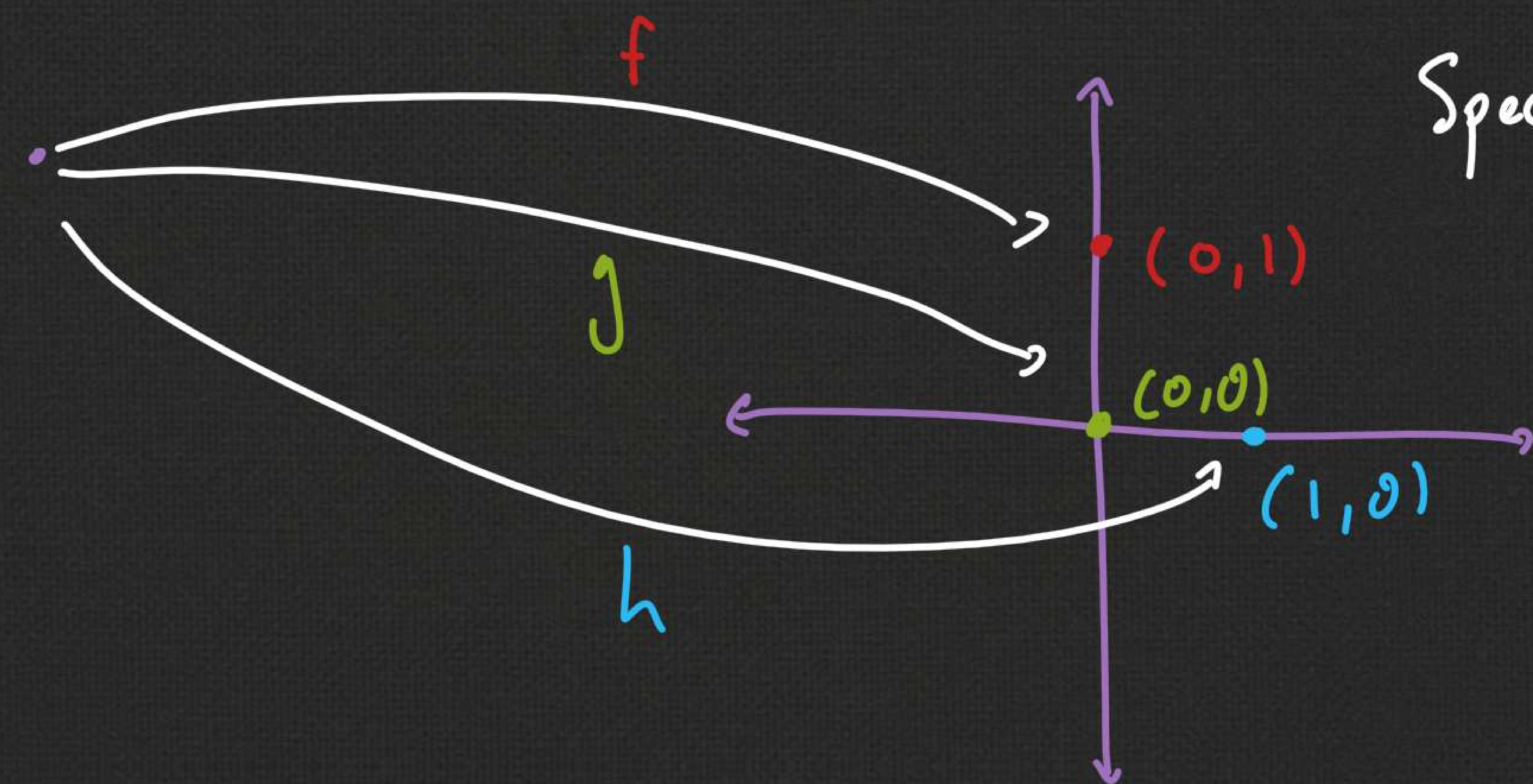
if $\exists\ h : X \times \mathbb{A}^1 \longrightarrow Y$ making the diagram

$$\begin{tikzcd}
X \cong X \times \{1\} \arrow[dr, "i_1"] \arrow[drr, bend left, "g"] & & \\
& X \times \mathbb{A}^1 \arrow[r, "h"] & Y \\
X \cong X \times \{0\} \arrow[ur, "i_0"] \arrow[urr, bend right, "f"'] & &
\end{tikzcd}$$

commutative.

This almost works. If $\sim$ is the relation that you get from this, then $\sim$ is reflexive and symmetric, but not transitive.

<u>Ex</u> $\quad X = \operatorname{Spec}(k) \qquad\qquad \operatorname{Spec}\left(k[x,y]/(xy)\right) = Y$



Here $f \sim g$, and $g \sim h$, but $f \not\sim h$ since any $k[x,y]/(xy) \to k[t]$ factors

$$\begin{tikzcd}
 & k[x] \\ (y=0) \arrow[dr] & \\
k[x,y]/(xy) \arrow[ur] \arrow[dr] & & k[t] \\
 & k[y] \\ (x=0) \arrow[ur] &
\end{tikzcd}$$

Def$^{\underline{n}}$: an <u>elementary finite correspondence from X to Y</u> is an irreducible closed subset of $X \times Y$ whose associated integral scheme is finite and surjective over X via the projection $X \times Y \to X$. (e.g. if $f: X \to Y$, the Γ_f)

More generally, <u>a finite correspondence from X to Y</u> is an element of the free abelian group generated by elementary finite correspondences <u>$Cor(X, Y)$</u>.

For X, Y, Z smooth there is a composition

$$\mathrm{Cor}(X,Y) \times \mathrm{Cor}(Y,Z) \longrightarrow \mathrm{Cor}(X,Z)$$
$$(\alpha, \beta) \longmapsto \beta \circ \alpha$$

defined on elementary finite correspondences $\alpha = [V]$, $\beta = [W]$ as the pushforward cycle (along $X \times Y \times Z \longrightarrow X \times Z$) of the intersection cycle $[V \times Z] \cap [X \times W]$.

(if $f : X \to Y$ and $g : Y \to Z$, then $[\Gamma_g] \circ [\Gamma_f] = [\Gamma_{g \circ f}]$)

$V \subset X \times Y$ and $W \subset Y \times Z$

$$V \times_Y W \longrightarrow (X \times Y) \times_Y (Y \times Z) = X \times Y \times Z$$

has image $(V \times Z) \cap (X \times W)$ and

$$\operatorname{codim}(V \times_Y W) = \dim(Z) + \dim(Y)$$
$$= \operatorname{codim}(X \times W) + \operatorname{codim}(V \times Z).$$

$$\begin{tikzcd}
V \times_Y W \arrow[r] \arrow[d, "\text{fin.}"'] \arrow[dr, phantom, "\ulcorner", very near start] & W \arrow[r] \arrow[d, "\text{fin.}"] & Z \\
V \arrow[r] \arrow[d, "\text{fin.}"'] & Y & \\
X & &
\end{tikzcd}$$

Then $[V \times Z] \cap [X \times W] = \sum_i m_P [P]$ where
P irred.

$$m_P = \sum_{i \geq 0} (-1)^i \operatorname{length}_{\mathcal{O}_{X \times Y \times Z, P}} \operatorname{Tor}_i^{\mathcal{O}_{X \times Y \times Z, P}}\left(\mathcal{O}_{V \times Z, P}, \mathcal{O}_{X \times W, P}\right).$$

$$\begin{tikzcd}
P \arrow[r, "\text{fin.}"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & V \times_Y W \arrow[d, "\text{fin.}"] \\
X \times Z \arrow[r] & X
\end{tikzcd}$$

So $P \to X \times Z$ is finite,
and $P \twoheadrightarrow X$.

or correspondences!

Def$\underline{\text{n}}$: a <u>naive $\mathbb{A}^1$-homotopy between morphisms</u> ✓ $X \overset{f\ \alpha}{\underset{g\ \beta}{\rightrightarrows}} Y$ is a finite correspondence $h \in Cor(X \times \mathbb{A}^1, Y)$ so that

$$\left(\overset{\alpha}{[\Gamma_f]} = h \circ [\Gamma_{i_0}] \quad \text{and} \quad \overset{\beta}{[\Gamma_g]} = h \circ [\Gamma_{i_1}] \right)$$

where $X \overset{i_0}{\underset{i_1}{\rightrightarrows}} X \times \mathbb{A}^1$ are the inclusions.

This is an equivalence relation (on $Hom(X,Y)$ or $Cor(X,Y)$) that we'll denote by $[\sim_{\mathbb{A}^1}]$

Now with a working notion of (naive) $\mathbb{A}^1$-homotopies, we can define a notion of naive $\mathbb{A}^1$-homotopy equivalences.

Def$^{\underline{n}}$: a morphism $f: X \to Y$ is a <u>naive $\mathbb{A}^1$-homotopy equivalence</u> if $\exists\, g: Y \to X$ so that $\Gamma_{g \circ f} \sim_{\mathbb{A}^1} \Delta_X$ and $\Gamma_{f \circ g} \sim_{\mathbb{A}^1} \Delta_Y$.

$$\Delta_X = \Gamma_{id_X} \subseteq X \times X$$

Question: What properties of a variety are preserved by naive $\mathbb{A}^1$-homotopy equivalences?

Part 2:

a "rigidity" type theorem

Theorem 1: if X, Y are smooth projective varieties, then X, Y are naively $\mathbb{A}^1$-homotopy equivalent if and only if $X \cong Y$.

In general, naive $\mathbb{A}^1$-homotopy equivalences are nontrivial.

Ex For any smooth X, the projection $X \times \mathbb{A}^1 \to X$ is a naive $\mathbb{A}^1$-homotopy equivalence.

Outline of Proof: (In a number of steps)

Step # 1: Show that two finite correspondences $\alpha, \beta \in \mathrm{Cor}(X,Y)$ (with X, Y proper) define the same cycle class in $CH(X \times Y)$ if $\alpha \sim_{\mathbb{A}^1} \beta$.

Main difficulty:

$$\begin{tikzcd}
\mathrm{Cor}(X \times \mathbb{A}^1, Y) \arrow[r, shift left, "\circ \Gamma_{i_0}"] \arrow[r, shift right, "\circ \Gamma_{i_1}"'] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Cor}(X, Y) \arrow[d] \\
CH(X \times \mathbb{A}^1 \times Y) \arrow[r, dashed, shift left, "\circ \Gamma_{i_0}"] \arrow[r, dashed, shift right, "\circ \Gamma_{i_1}"'] & CH(X \times Y)
\end{tikzcd}$$

Two Solutions

#1: lift from $\mathbb{A}^1$ to $\mathbb{P}^1$

#2: show this diagram exists with Gysin pullbacks

Step #2: any cycle class $\alpha \in CH(X \times Y)$ defines a pushforward

$$\alpha_\cdot : CH(X) \longrightarrow CH(Y)$$
$$\beta \longmapsto \pi_{Y\cdot}(\alpha \cap \pi_X^{-1}(\beta))$$

$$\begin{tikzcd} & X \times Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X & & Y \end{tikzcd}$$

So combined with Step #1, we get that if $\alpha \sim_{\mathbb{A}^1} \beta$ then $\alpha_\cdot = \beta_\cdot : CH(X) \longrightarrow CH(Y)$.

(For naive $\mathbb{A}^1$-homotopy equivalences $\Delta_X \sim_{\mathbb{A}^1} \Gamma_{g \circ f}$ and $\Delta_Y \sim_{\mathbb{A}^1} \Gamma_{f \circ g}$ says $\underline{\Delta_{X\cdot}} = id_{CH(X)} = [\Gamma_{g\circ f}]_\cdot = (g \circ f)_\cdot = \underline{g_\cdot \circ f_\cdot}$ and $\underline{\Delta_{Y\cdot}} = id_{CH(Y)} = \underline{f_\cdot} \circ g_\cdot$.)

Ex Let $\mathbb{P}^1_{x,y}$ and $\mathbb{P}^2_{X,Y,Z}$. Then the map

$$h(t) : \mathbb{P}^1 \times \mathbb{A}^1 \longrightarrow \mathbb{P}^2$$

$$([x:y], t) \longmapsto [x^2 : txy : y^2]$$

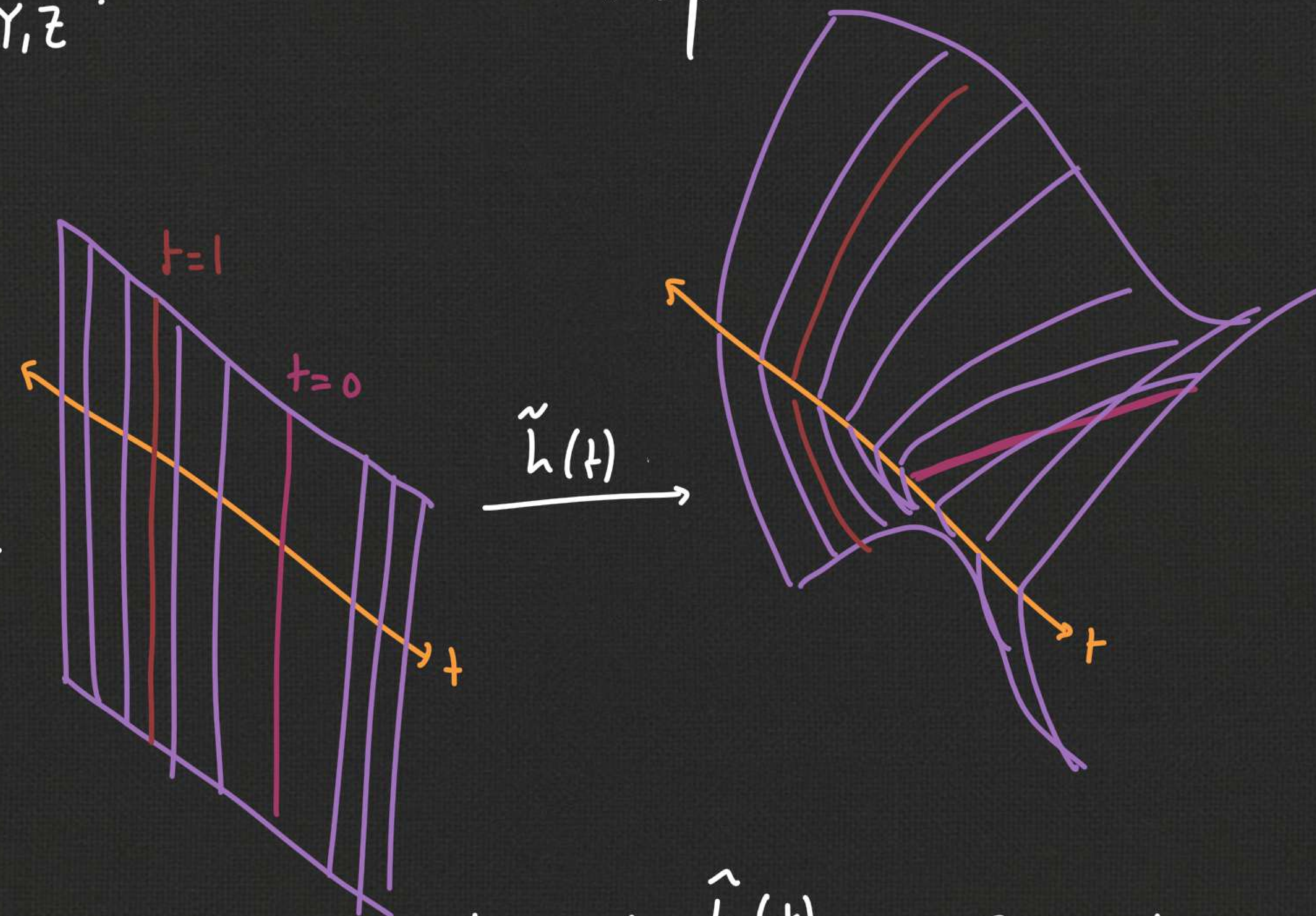
$$h(0)_*, h(1)_* : \underset{\mathbb{Z}}{\underset{\|}{CH_0(\mathbb{P}^1)}} \xrightarrow{\sim} \underset{\mathbb{Z}}{\underset{\|}{CH_0(\mathbb{P}^2)}}$$

$$h(0)_* : CH_1(\mathbb{P}^1) \longrightarrow CH_1(\mathbb{P}^2)$$

$$[\mathbb{P}^1] \longmapsto 2[Y=0]$$

$$h(1)_* : CH_1(\mathbb{P}^1) \longrightarrow CH_1(\mathbb{P}^2)$$

$$[\mathbb{P}^1] \longmapsto [XZ = Y^2]$$



$$\mathbb{P}^1 \times \mathbb{A}^1 \xrightarrow{\hat{h}(t)} \mathbb{P}^2 \times \mathbb{A}^1$$

$$(p,t) \longmapsto (h(t)(p), t)$$

Step #3: Prove a (strong) analog of a theorem of Whitehead:

Theorem 2: Let $f: X \to Y$ be a morphism between projective varieties X and Y such that

$$f_*: CH(X) \longrightarrow CH(Y)$$

is an isomorphism. If Y is normal, then f is an isomorphism.

(Altogether, this finishes the proof.) $\square$

Part 3:

analogs of Whitehead's theorem

Whitehead's theorem (together with Hurewicz's theorem) says:

if $f: X \to Y$ is a continuous map between simply connected CW complexes with $f_*: H_*(X, \mathbb{Z}) \to H_*(Y, \mathbb{Z})$ an isomorphism, then f is a homotopy equivalence.

We prove a few variants of the previous theorem also (in each case by reducing to Zariski's Main Theorem).

Theorem^{H}: if $f : X \to Y$ is a morphism (over $k = \mathbb{C}$) with X, Y projective smooth and $f_* : H_{2*}(X(\mathbb{C}), \mathbb{Z}) \longrightarrow H_{2*}(Y(\mathbb{C}), \mathbb{Z})$ is an iso. then so is f.

Theorem^{K}: if $f : X \to Y$ is a morphism with X, Y projective, Y normal, and $f_* : G(X) \longrightarrow G(Y)$ an iso. then so is f.

Theorem$^{\Omega}$: if $f : X \to Y$ is a morphism with X, Y projective, Y normal, char$(k) = 0$, and if $f_* : \Omega_*(X) \longrightarrow \Omega_*(Y)$ is an iso. then so is f.

These theorems are sharp.

Ex (1) (Can't drop normality) If $C = \mathrm{Proj}\left(k[x,y,z]/(y^2z = x^3)\right)$ then

the normalization $\nu : \mathbb{P}^1 \to C$ induces an isomorphism

$$\begin{array}{cccc} \nu_\cdot : & CH(\mathbb{P}^1) & \longrightarrow & CH(C) . \\ & \| & & \| \\ & \mathbb{Z}\oplus\mathbb{Z} & & \mathbb{Z}\oplus\mathbb{Z} \end{array}$$

(2) (Can't use $\mathbb{Q}$ coeff.) If Q is a 3 dim'l quadric and $k = \bar{k}$

then there is a finite map $Q \to \mathbb{P}^3$ with

$$\mathbb{Q}^{\oplus 4} = CH(Q)_{\mathbb{Q}} \to CH(\mathbb{P}^3)_{\mathbb{Q}} = \mathbb{Q}^{\oplus 4} \quad \text{an isomorphism.}$$

Lemma: Let $f : X \to Y$ be a morphism with X, Y projective and assume $f_* : CH(X) \to CH(Y)$ has torsion kernel. Then f is finite.

Proof. Assume f is not finite. Let $V \subset X$ have $\dim(V) > \dim(f(V)) \geq 0$. Then for any embedding $i : X \subset \mathbb{P}^n$ we get

$$\begin{tikzcd} CH(X) \arrow[r, "i_*"] \arrow[d, "f_*"'] & CH(\mathbb{P}^n) \\ CH(Y) & \end{tikzcd} \qquad \begin{tikzcd} {[V]} \arrow[r, mapsto] \arrow[d, mapsto] & \deg(V)[\mathbb{P}^{\overset{\dim(V)}{\overset{\shortparallel}{m}}}] \neq 0 \\ 0 & \end{tikzcd}$$

$\not\square$

Lemma: Let $f: X \to Y$ be a morphism with X and Y projective. Assume that $f_*: CH(X) \to CH(Y)$ is surjective. Then

(1) f is surjective

(2) if f is finite, then f is birational.

Proof. For (1), let $j: U \subset Y$ be the inclusion of an open subscheme. Then

$$\begin{tikzcd} f^{-1}(U) \arrow[r, "\hat{j}"] \arrow[d, "\hat{f}"'] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d, "f"] \\ U \arrow[r, "j"] & Y \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} CH(f^{-1}(U)) \arrow[d, two heads, "\hat{f}_*"'] & CH(X) \arrow[l, "\hat{j}^*"'] \arrow[d, two heads, "f_*"] \\ CH(U) & CH(Y) \arrow[l, two heads, "j^*"'] \end{tikzcd}$$

Taking $U = Y \setminus f(X)$ shows $U = \emptyset$.

For (2), note

$$f_*: \mathbb{Z} = CH^0(X) \twoheadrightarrow CH^0(Y) = \mathbb{Z}$$
$$[X] \longmapsto \deg(f) \cdot [f(X)].$$

$\square$

Altogether, if $f_* : CH(X) \longrightarrow CH(Y)$ is an isomorphism, then f is both finite and birational. If Y is normal, then Zariski's Main Theorem implies f is an isomorphism. $\square$


THANKS!